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A CONVERGENCE PROBLEM CONNECTED WITH CONTINUED FRACTIONS

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ABSTRACT. The set $Z_\alpha := \{\beta \mid \lim_{n\to\infty} \|q_n\beta\| = 0\}$ is considered, where $(q_n)_{n\in\mathbf{N}}$ is the sequence of best approximation denominators of α, and it is explicitly determined for α with bounded continued fraction coefficients.

Introduction. Let the irrational number α have continued fraction expansion $\alpha = [a_0; a_1, a_2, a_3, \ldots]$ and let $q_{-1} = 0$, $1 = q_0 \leq q_1 < q_2 < \cdots$ with $q_{i+1} = a_{i+1}q_i + q_{i-1}$ be the best approximation denominators of α. By Theorem 4.3 in [**2**] it follows, that $\{q_n\beta\}_{n\in\mathbf{N}}$ (where $\{x\} := x - [x]$ denotes the fractional part of x) is uniformly distributed modulo one for almost all $\beta \in \mathbf{R}$ in the sense of Lebesgue measure. Clearly the sequence is not uniformly distributed for all β, since if we take $\beta = m\alpha + n$ with $m, n \in \mathbf{Z}$, then $\lim_{n\to\infty} \|q_n\beta\| = 0$ where $\|x\|$ denotes the distance from x to the nearest integer.

We ask now for the set $Z_\alpha := \{\beta \in \mathbf{R} \mid \lim_{n\to\infty} \|q_n\beta\| = 0\}$. The problem of determining this set also is of some importance in some problems of automata theory [**3**]. We will show

THEOREM 1. *If α has bounded continued fraction coefficients, then*

$$\lim_{n\to\infty} \|q_n\beta\| = 0$$

if and only if $\beta = m\alpha + n$ with $m, n \in \mathbf{Z}$.

This in general is not true if α does not have bounded continued fraction coefficients, for we can show

THEOREM 2. *There are $\alpha \in \mathbf{R}$ and $\beta \neq m\alpha + n$ for all $m, n \in \mathbf{Z}$ with*

$$\lim_{n\to\infty} \|q_n\beta\| = 0.$$

PROOFS. For the proof of Theorem 1 we need two lemmata. In these two lemmata let α be a fixed real number and the q_i as above.

LEMMA 1. *Let $a_i \leq K - 1$ for all $i \in \mathbf{N}$ and let $n, p_n, p_{n+1} \in \mathbf{N}$ be given, then there exists at most one $\beta \in [0, 1)$ with*

$$|q_n\beta - p_n| \leq \frac{1}{4K}, \quad |q_{n+1}\beta - p_{n+1}| \leq \frac{1}{4K}$$

and

$$\|q_m\beta\| \leq \frac{1}{4K} \quad \textit{for all } m \geq n.$$

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PROOF. For any β which fulfills our conditions, we have

$$q_{n+2}\beta \in I_{n+2} := \left(p_{n+1}\cdot\frac{q_{n+2}}{q_{n+1}} - \frac{q_{n+2}/q_{n+1}}{4K}, p_{n+1}\cdot\frac{q_{n+2}}{q_{n+1}} + \frac{q_{n+2}/q_{n+1}}{4K}\right).$$

The length of I_{n+2} is $\leq \frac{1}{2}$ and because of $\|q_{n+2}\beta\| \leq 1/4K$, there is, independent of β, exactly one p_{n+2} with $|q_{n+2}\beta - p_{n+2}| \leq 1/4K$. Going on this way, we see, that independent of β, by n, p_n and p_{n+1} all further values $p_{n+2}, p_{n+3}, \ldots$ are determined, for which $|q_m\beta - p_m| \leq 1/4K$. Therefore β is unique, because $\beta = \lim_{m\to\infty}(p_m/q_m)$.

LEMMA 2. *Let β, p_n and p_{n+1} be such that $|q_n\beta - p_n| < 1/8K$, $|q_{n+1}\beta - p_{n+1}| < 1/8K$, $p_n/q_n \neq p_{n+1}/q_{n+1}$, $\operatorname{sgn}(q_n\beta - p_n) = -\operatorname{sgn}(q_{n+1}\beta - p_{n+1})$ and $\|q_m\beta\| < 1/4K$ for $m \geq n$; if we define $\lambda_n := [0; a_n, a_{n+1}, \ldots]$ and $\alpha_0 := (p_{n+1} + \lambda_{n+1}p_n)/(q_{n+1} + \lambda_{n+1}q_n)$, then $\beta = \alpha_0$.*

PROOF. We show $|q_n\alpha_0 - p_n| < 1/4K$, $|q_{n+1}\alpha_0 - p_{n+1}| < 1/4K$ and $\|q_m\alpha_0\| < 1/4K$ for $m \geq n$ and then by Lemma 1 the result follows. We have

$$\left|q_n\cdot\frac{p_{n+1}+\lambda_{n+1}p_n}{q_{n+1}+\lambda_{n+1}q_n} - p_n\right| = \left|\frac{q_np_{n+1} - q_{n+1}p_n}{q_{n+1}+\lambda_{n+1}q_n}\right| \leq \left|q_n\cdot\frac{p_{n+1}}{q_{n+1}} - p_n\right|$$
$$\leq |q_n\beta - p_n| + \left|q_n\cdot\frac{1}{8Kq_{n+1}}\right| \leq \frac{1}{4K}$$

and

$$\left|q_{n+1}\cdot\frac{p_{n+1}+\lambda_{n+1}p_n}{q_{n+1}+\lambda_{n+1}q_n} - p_{n+1}\right| = \left|\lambda_{n+1}\cdot\frac{p_nq_{n+1} - q_np_{n+1}}{q_{n+1}+\lambda_{n+1}q_n}\right| \leq \frac{1}{4K}.$$

Now we consider in the plane the lattice Γ produced by the vectors (q_n, p_n) and (q_{n+1}, p_{n+1}) which are independent, and the line

$$g:\quad y = \frac{p_{n+1}+\lambda_{n+1}p_n}{q_{n+1}+\lambda_{n+1}q_n}\cdot x.$$

This lattice is isomorphic to the lattice Γ' produced by $(0,1)$, $(1,0)$ and the line $g'\colon\ y = \lambda_{n+1}\cdot x$.

Now we consider the Klein-model of the development of λ_{n+1} to continued fractions as it is explained for example in [**1**]. Γ' is the lattice for this development.

By observing the same operations in Γ as in Γ' according to this development we see that

$$\|q_m\alpha_0\| \leq \max(\|q_n\alpha_0\|, \|q_{n+1}\alpha_0\|) < \frac{1}{4K} \quad \text{for all } m \geq n.$$

(See Figure 1 and Figure 2.)

For example we have (see Figure 3) $\|q_{n+3}\alpha_0\| = |R_2T_2|$ and since $|P_2S_2'| \leq \max(|P_{-1}0|, |P_0S_0'|)$ we have $|R_2S_2| \leq \max(|R_{-1}0|, |R_0S_0|)$ and therefore

$$|R_2T_2| \leq \max(|R_{-1}T_0|, |R_0T_1|) = \max(\|q_n\alpha_0\|, \|q_{n+1}\alpha_0\|),$$

and the result follows.

PROOF OF THEOREM 1. Let $\lim_n \|q_n\beta\| = 0$. Of course β must be irrational or an integer. So let β be irrational. We define by p_n the integer lying next to $q_n\beta$, and N_0 to be such, that $\|q_n\beta\| < 1/8K$ for $n \geq N_0$.

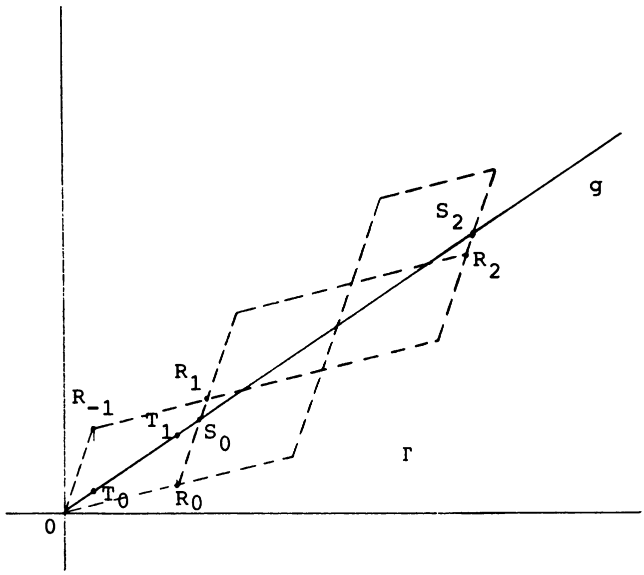

FIGURE 1

We show, that there is a $n \geq N_0$ such, that $\operatorname{sgn}(q_n\beta - p_n) = -\operatorname{sgn}(q_{n+1}\beta - p_{n+1})$. Since $\beta \notin \mathbf{Q}$, from this it follows, that $p_n/q_n \neq p_{n+1}/q_{n+1}$.

If for example $q_n\beta - p_n > 0$ for all $n \geq N_0$, then for all $n \geq N_0 + 1$ we would have $q_{n+1}\beta = a_n q_n \beta + q_{n-1}\beta$. Further $q_n\beta = p_n + \varepsilon_0$, $q_{n-1}\beta = p_{n-1} + \varepsilon_1$ with $0 < \varepsilon_i < 1/8K$ and therefore $q_{n+1}\beta = a_n p_n + p_{n-1} + a_n\varepsilon_0 + \varepsilon_1$ with $0 < a_n\varepsilon_0 + \varepsilon_1 < 1/8 + 1/8K \leq 1/4$ and therefore $\|q_{n+1}\beta\| = a_n\varepsilon_0 + \varepsilon_1 \geq \|q_n\beta\|$ for all $n \geq N_0 + 1$ and this is a contradiction since $\|q_{N_0+1}\beta\| \neq 0$. So by Lemma 2

$$\beta = \frac{p_{n+1} + \lambda_{n+1} p_n}{q_{n+1} + \lambda_{n+1} q_n}.$$

If we define Q_i such, that $|q_i\alpha - Q_i| \leq 1/q_{i+1}$ for all i, then (see [**4**]).

$$\lambda_{n+1} = [0; a_{n+1}, a_{n+2}, \dots] = -\frac{Q_{n+1} - \alpha q_{n+1}}{Q_n - \alpha q_n}$$

and therefore

$$\beta = \frac{(p_{n+1}Q_n - Q_{n+1}p_n) + \alpha(q_{n+1}p_n - q_n p_{n+1})}{(q_{n+1}Q_n - Q_{n+1}q_n)} = a\alpha + b \quad \text{with } a, b \in \mathbf{Z}$$

because $|q_{n+1}Q_n - Q_{n+1}q_n| = 1$, and the theorem is proved.

PROOF OF THEOREM 2. Let γ and δ irrational be such that

$$\delta \neq (a\gamma + b)/(c\gamma + d) \qquad \text{for all } a, b, c, d \in \mathbf{Z}$$

and such that $1, \gamma, \delta$ are linear independent over $\mathbf{Q}$. Therefore not both are of the form $e\alpha + f$ with $e, f \in \mathbf{Z}$. We define now a real number α by its partial quotients $a_0, a_1, \dots$ such that for the best approximation denominators q_n of α we have $\lim_{n\to\infty} \|q_n\gamma\| = 0$ and $\lim_{n\to\infty} \|q_n\delta\| = 0$.

Let $a_0 = 0$, $q_{-1} = 0$, $q_0 = 1$ and assume that $a_1, a_2, \dots, a_n$ and therefore $q_1, q_2, \dots, q_n$ are defined.

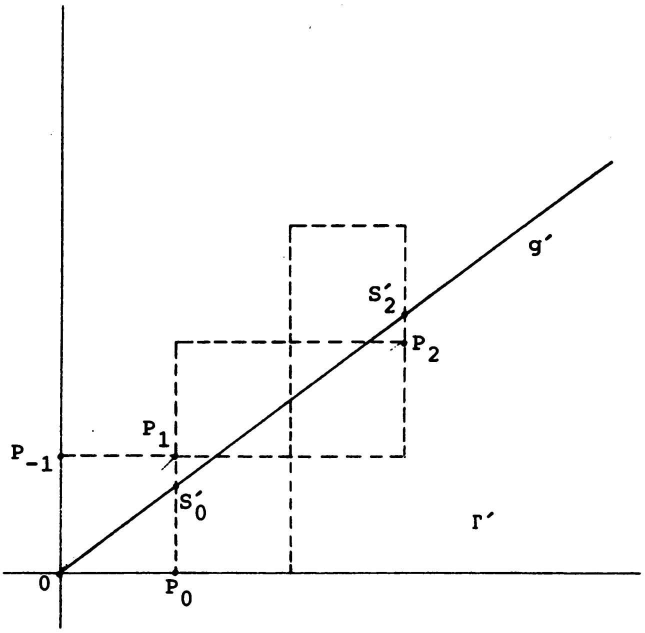

FIGURE 2

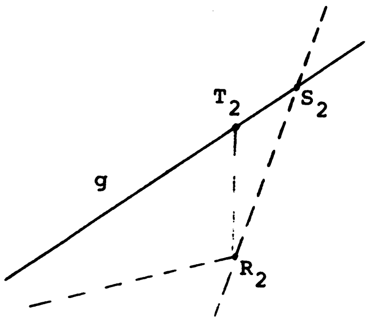

FIGURE 3

$1, q_n\gamma$ and $q_n\delta$ are linearly independent over $\mathbf{Q}$ and therefore the sequence $(\{kq_n\gamma\}, \{kq_n\delta\})_{k\in\mathbf{N}}$ is dense in the unit square. Therefore we can choose an integer a_{n+1} such that

$$\max(\|a_{n+1}q_n\gamma + q_{n-1}\gamma\|, \|a_{n+1}q_n\delta + q_{n-1}\delta\|) \le 1/n.$$

Then with $q_{n+1} = a_{n+1}q_n + q_{n-1}$ we have $\lim_{n\to\infty} \|q_n\beta\| = 0$ for $\beta = \gamma$ and for $\beta = \delta$ and the theorem is proved.

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