NEAR-FIELDS ASSOCIATED WITH INVARIANT LINEAR $\kappa$-RELATIONS

PETER FUCHS AND C. J. MAXSON

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ABSTRACT. In this paper we investigate a construction method for subnear-rings of $M(G)$ proposed by H. Wielandt using subgroups of direct powers $G^\kappa$ of $G$ called invariant linear $\kappa$-relations. If $\kappa = 2$ we characterize, in terms of properties of these subgroups, when the associated near-rings are near-fields and prove that every near-field arising from an invariant linear 2-relation must be a field.

I. Introduction. In 1972 H. Wielandt [7] presented a very general method for constructing subnear-rings of the near-ring $M(G)$ of functions on the group $G$. A particular instance of this construction, namely centralizer near-rings, has been extensively investigated in the past several years. In this paper we initiate a study of the structure of the near-rings obtained by Wielandt's general method.

We recall the construction. Let $(G, +)$ be a group, let $\kappa$ be a cardinal number and let $G^\kappa$ denote the direct product of $\kappa$ copies of $G$. We let $M(G)$ act on $G^\kappa$ component-wise. For any subgroup $H$ of $G^\kappa$ we define

$$M(G, \kappa, H) = \{ f \in M(G) | f(H) \subseteq H \}.$$ 

These $M(G, \kappa, H)$ are subnear-rings of $M(G)$ with identity $id : G \rightarrow G$, $id(x) = x \forall x \in G$.

One is therefore led to an investigation of the transfer of information between the structure of the near-rings $M(G, \kappa, H)$ and the subgroups $H$ of $G^\kappa$. Wielandt calls these subgroups invariant linear $\kappa$-relations and indicates that these linear $\kappa$-relations might be studied as in his work on permutation groups $P$ via $P$-invariant $\kappa$-relations [6].

Another reason for investigating the near-rings $M(G, \kappa, H)$ is that they are indeed very general as indicated in the following theorem. Let $R$ be a near-ring with identity, 1. It is well known that $R$ can be embedded in $M(G)$ for some group $G$.

THEOREM I.1. Let $R$ be a near-ring with identity 1. Then there exists a group $G$, a cardinal number $\kappa$, and a subgroup $H$ of $G^\kappa$ such that $R \cong M(G, \kappa, H)$.

The reader is referred to the books by Meldrum [2] and Pilz [3] for the proof of this result as well as background information on near-rings.

In [4], Remak investigated the subgroup structure of $G^2$ and in [5] indicated how this can be extended to the case $\kappa \geq 3$. We briefly outline his results. Again let $G$ be a group, $\kappa$ a positive integer, $\kappa \geq 2$, and for $j \in \{1, \ldots, \kappa\}$ let $B_j$ be a...
subgroup of \( G \), \( \overline{B}_j \) a normal subgroup of \( B_j \) such that \( B_j/\overline{B}_j \simeq B_{j+1}/\overline{B}_{j+1} \) with isomorphisms \( \sigma_j, j \in \{1, \ldots, \kappa-1\} \). Let \( \alpha \) be an ordinal, \( \{ b_{1\eta} | \eta < \alpha \} \) a set of coset representatives of \( \overline{B}_1 \) in \( B_1 \) where \( b_{10} = 0 \) and define a subset \( H \subseteq G^\kappa \) by

\[
H = \bigcup_{\eta < \alpha} \left( (b_{1\eta} + \overline{B}_1) \times \prod_{j=1}^{\kappa-1} (\sigma_j \circ \sigma_{j-1} \circ \cdots \circ \sigma_1 (b_{1\eta} + \overline{B}_1)) \right).
\]

\( H \) is called a \( \kappa \)-fold meromorphic product and will be denoted by

\[
H = \overline{B}_1/\sigma_1 B_2/\sigma_2 \cdots /\sigma_{\kappa-1} B_\kappa/\overline{B}_\kappa.
\]

It is straightforward to verify that \( H \) is a subgroup of \( G^\kappa \) but in general not every subgroup of \( G^\kappa \) is a \( \kappa \)-fold meromorphic product. For \( \kappa = 2 \), however we have such a result.

**Theorem 1.2 (Klein-Fricke) [4].** Every subgroup of \( G \times G \) is a 2-fold meromorphic product.

In this paper we focus on near-fields for the case \( \kappa = 2 \). In the next section we characterize when \( M(G, 2, H) \) is a near-field and find the somewhat surprising result that the only near-fields arising in this case are fields.

**II. When is \( M(G, 2, H) \) a near-field?** We now turn to a characterization of the triples \((G, 2, H)\) such that \( M(G, 2, H) \) is a near-field. From the Klein-Fricke Theorem we know that \( H = B_1/\overline{B}_1 \times B_2/\overline{B}_2 \). For \( G = \mathbb{Z}_2 \) the subgroups \( H_1 = \mathbb{Z}_2/\mathbb{Z}_2 \times \{0\}/\{0\} = \mathbb{Z}_2 \times \{0\}, H_2 = \{0\} \times \mathbb{Z}_2 \) and \( H_3 = \{0\} \times \{0\} \) are such that \( M(G, 2, H_i) \simeq \mathbb{Z}_2, i = 1, 2, 3 \). For \( H_4 = \{(0,0), (1,1)\} \) and \( H_5 = \mathbb{Z}_2 \times \mathbb{Z}_2 \) we get \( M(G, 2, H_4) = M(G, 2, H_5) = M(\mathbb{Z}_2) \) which is not a near-field. For the remainder of the paper we take \( |G| > 2 \) and in a sequence of lemmas show that when \( M(G, 2, H) \) is a near-field, \( H \) has the form \( G \times \sigma G \). For a subgroup \( S \) of \( G \) we let \( S^* \) denote \( S \setminus \{0\} \).

**Lemma II.1.** Let \( H = B_1/\overline{B}_1 \times B_2/\overline{B}_2 \). If \( N = M(G, 2, H) \) is a near-field then \( B_1 = B_2 = G \).

**Proof.** We may assume that \( B_1 \cup B_2 \neq \{0\} \) since otherwise \( M(G, 2, H) = M_0(G) \) is not a near-field. If \( B_1 \cup B_2 \neq G \) then the function \( f : G \to G \) given by \( f(x) = x \) if \( x \in G \setminus (B_1 \cup B_2) \) and \( f(x) = 0 \) if \( x \in B_1 \cup B_2 \) is in \( N \) contradicting the fact that \( N \) is a near-field. Hence \( B_1 \cup B_2 = G \) so at least one of \( B_1, B_2 \) must equal \( G \), say \( B_1 = G \). Suppose \( B_2 \neq G \) and take \( y \in G \setminus B_2 \).

**Case (i).** \( \overline{B}_1 \neq \{0\} \). Let \( b_1 \in \overline{B}_1 \). One verifies that the function \( h : h(y) = b_1 \) and \( h(x) = 0 \) for \( x \neq y \) is in \( N \). Since \( |G| \geq 3 \), \( h \) is not invertible, a contradiction.

**Case (ii).** \( \overline{B}_1 = \{0\} \). Then \( H = G/\{0\} \times B_2/\overline{B}_2 \). Now \( \sigma(y) = b_2 + \overline{B}_2 \) for some \( b_2 \in B_2 \setminus \overline{B}_2 \). Define \( A_1 = \{ x, x \in b_2 + \overline{B}_2 \}, A_\eta = \bigcup \{ \sigma(x) | x \in A_{\eta-1} \} \) for \( n \geq 2 \). Let \( A = \bigcup_{n=1}^\infty A_\eta \). We define \( f : G \to G \) by \( f(x) = 0, x \in A \cup \{y\} \) and \( f(x) = x, x \notin A \cup \{y\} \) and note that \( f \in N \). If \( 0 \neq y' = y + b_2 \), then \( y' \notin A \cup \{y\} \) since \( A \subseteq B_2 \). Hence \( f(y') = y' \) so \( f \) is not the zero map. Since \( f \) is not invertible, we have a contradiction.

Therefore we must conclude that \( B_1 = B_2 = G \).

Now let \( H = G/B_1 \times \sigma G/B_2 \) and let \( N = M(G, 2, H) \). When \( N \) is a near-field, \( N \) is zero-symmetric since \( N \) contains the identity map and therefore cannot be
isomorphic to the constant maps on $\mathbb{Z}_2$. Thus in this case we must have $B_1 \cap B_2 = \{0\}$, for if $0 \neq y \in B_1 \cap B_2$ then $f: G \to G$, $f(x) = y \forall x \in G$ is an element of $N_G \setminus \{0\}$, a contradiction. Thus $B_1 \oplus B_2$ is a normal subgroup of $G$. We now develop some notation. Let $\alpha, \beta$ be ordinals such that $B_1 = \{b_{1\eta} | \eta < \alpha\}$, $B_2 = \{b_{2\gamma} | \gamma < \beta\}$ where $b_{10} = b_{20} = 0$. Further, if $B_1 \neq \{0\}$, let $\sigma(y + B_1) = b_{1\eta} + B_2$, $1 \leq \eta < \alpha$ and $\sigma(b_{2\gamma} + B_1) = x_\gamma + B_2$, $1 \leq \gamma < \beta$ if $B_2 \neq \{0\}$. Define $X_0 = \emptyset$, $X_n = \{x + B_2 | 1 \leq \gamma < \beta\}$ and for $n \geq 1$, $X_n = \{x + B_1 | x + B_1 \cap w + B_2 \neq \emptyset \text{ for some } w + B_2 \in X_{n-1}\}$, and $\bar{X}_n = \{\sigma(x + B_1) | x + B_1 \in X_n\}$. Then define $X = \bigcup_{n=0}^{\infty} (\bigcup X_n \cup \bigcup \bar{X}_n)$. If $B_2 = \{0\}$ define $X = \emptyset$. In a similar manner let $Y_0 = \emptyset$, $Y_n = \{y + B_1 | 1 \leq \eta < \alpha\}$, and for $n \geq 1$, $Y_n = \{y + B_2 | y + B_2 \cap w + B_1 \neq \emptyset \text{ for some } w + B_1 \in Y_{n-1}\}$, $\bar{Y}_n = \{\sigma^{-1}(y + B_2) | y + B_2 \in Y_n\}$. Let $Y = \bigcup_{n=0}^{\infty} (\bigcup Y_n \cup \bigcup \bar{Y}_n)$. If $B_1 = \{0\}$ define $Y = \emptyset$. For Lemmas II.2–II.5 we always assume that $B_1 \neq \{0\}$ and $B_2 \neq \{0\}$. If either $B_1 = \{0\}$ or $B_2 = \{0\}$ then the results are trivial or can be seen to hold by making obvious modifications.

**Lemma II.2.** For $n \geq 0$ let $A_n = \bigcup_{k=0}^{n} \bigcup \bar{Y}_{k} \cup B_1$ and $G_n = \bigcup_{k=0}^{n} \bigcup \bar{X}_{k} \cup B_2$. Then $A_n$ and $G_n$ are subgroups of $G$ for each $n \geq 0$.

**Proof.** We show that $A_n$ is a subgroup of $G$ for $n \geq 0$. A similar argument can be used for $G_n$, $n \geq 0$. Since $B_1 \oplus B_2 = B_2 \oplus B_1$ is a subgroup of $G$, so is $\bigcup \bar{Y}_0 \cup B_1$. Suppose we have shown that $A_m$ is a subgroup for all $0 < m < n$. Let $z_1, z_2 \in A_n$.

Case (i). If $z_1, z_2 \in A_{n-1}$, then $z_1 - z_2 \in A_{n-1} \subseteq A_n$.

Case (ii). If $z_1, z_2 \in \bigcup \bar{Y}_n$ let $w_1, w_2 \in \bigcup Y_n$ such that $w_1, w_2 \in \bigcup \bar{Y}_{n-1}$ with $(z_1, w_1) \in H$, $(z_2, w_2) \in H$. Thus $(z_1 - z_2, w_1 - w_2) \in H$. If $w_1 - w_2 = 0$, then $z_1 - z_2 \in B_1$. If $w_1 - w_2 \in B_1^*$, then $z_1 - z_2 \in \bigcup \bar{Y}_0$. Finally, if $w_1 - w_2 \in \bigcup \bar{Y}_i$ for some $i \leq n - 1$, then $w_1 - w_2 \in \bigcup Y_{i+1}$, thus $z_1 - z_2 \in \bigcup \bar{Y}_{i+1} \subseteq A_n$.

Case (iii). If $z_1 \in B_1$, $z_2 \in \bigcup \bar{Y}_n$, let $y \in \bigcup Y_n$ such that $y \in \bigcup \bar{Y}_{n-1}$ with $(z_2, y) \in H$. Since $(z_1, 0) \in H$, $(z_1 - z_2, -y) \in H$. Now $-y \in \bigcup \bar{Y}_i$ for some $i \leq n - 1$, hence $y \in \bigcup Y_{i+1}$ and $z_1 - z_2 \in \bigcup \bar{Y}_{i+1} \subseteq A_n$.

Case (iv). If $z_1 \in \bigcup \bar{Y}_n$, $z_2 \in \bigcup \bar{Y}_0$, let $w \in \bigcup \bar{Y}_{n-1}$ and $b_1 \in B_1^*$ such that $(z_1, w) \in H$, $(z_2, b_1) \in H$. Thus $(z_1 - z_2, w - b_1) \in H$. Since $w - b_1 \in \bigcup \bar{Y}_{n-1}$, $w - b_1 \in \bigcup Y_n$, thus $z_1 - z_2 \in \bigcup \bar{Y}_n \subseteq A_n$.

Case (v). Finally let $z_1 \in \bigcup \bar{Y}_i$ for some $n > i > 0$ and $z_2 \in \bigcup \bar{Y}_n$. Let $w_1 \in \bigcup \bar{Y}_{i-1}$, $w_2 \in \bigcup \bar{Y}_{n-1}$ such that $(z_1, w_1) \in H$, $(z_2, w_2) \in H$. Thus $(z_1 - z_2, w_1 - w_2) \in H$. Since $w_1 - w_2 \in \bigcup \bar{Y}_{n-1}$, $w_1 - w_2 \in B_1$ or $w_1 - w_2 \in \bigcup \bar{Y}_i$ for some $i \leq n - 1$. If $w_1 - w_2 = 0$ then $z_1 - z_2 \in B_1 \subseteq A_n$, if $w_1 - w_2 \in B_1^*$ then $z_1 - z_2 \in \bigcup \bar{Y}_0 \subseteq A_n$. If $w_1 - w_2 \in \bigcup \bar{Y}_i$, then $w_1 - w_2 \in \bigcup \bar{Y}_{i+1}$, so $z_1 - z_2 \in \bigcup \bar{Y}_{i+1} \subseteq A_n$.

**Lemma II.3.** If $N = M(G, 2, H)$ is zero-symmetric then $Y \cap X = \emptyset$.

**Proof.** We first show that $\bigcup \bar{Y}_0 \cap X = \emptyset$. Suppose $y \in \bigcup \bar{Y}_0 \cap B_2$. Let $b_1 \in B_1^*$ such that $(y, b_1) \in H$. Since $y \in B_2$, $(b_1, y) \in H$. Thus $(y + b_1, b_1 + y) = (b_1 + y, b_1 + y) \in H$. Since $N$ is zero-symmetric $b_1 + y = 0$, so $y = -b_1$. But $B_1 \cap B_2 = \{0\}$, hence $b_1 = 0$, a contradiction. Consequently $\bigcup \bar{Y}_0 \cap B_2 = \emptyset$. Suppose that $y \in \bigcup \bar{Y}_0 \cap \bigcup \bar{X}_0$. Let $b_1 \in B_1^*$, $b_2 \in B_2^*$ such that $(y, b_1) \in H$, $(b_2, y) \in H$. Then $(y + b_2, b_1 + y) \in H$ and $(y + b_2, b_1 + y) = (y + b_2, y + b_1)$ for some $b_1 \in B_1^*$. Since $(b_1, b_2) \in H$, $(y + b_2, b_1 + y + b_1 + b_2) = (y + b_1 + b_2, y + b_1 + b_2) \in H$. Director or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Thus \( y + b_1 + b_2 = 0 \) and \( y + b_1 = -b_2 \). Hence \( y + b_1 \in \bigcup Y_0 \cap B_2 \), a contradiction to our first statement. Therefore \( \bigcup Y_0 \cap G_0 = \emptyset \). Suppose that \( \bigcup Y_0 \cap G_m = \emptyset \) for all \( 0 \leq m < n \) and \( y \in \bigcup Y_0 \cap G_n \). Then \( y \in \bigcup X_n \) and there exists \( r_1, \ldots, r_m \in G_{n-1} \) and \( b_2 \in B_2 \) such that \( (r_1, y) \in H, (r_2, r_1) \in H, \ldots, (r_m, r_{m-1}) \in H \) and \( (b_2, r_m) \in H \). Therefore \( (r_1 + \cdots + r_m + b_2, y + r_1 + \cdots + r_m) \in H \). Since \( (y, b_1) \in H \) for some \( b_1 \in B_1 \), \( (y + r_1 + \cdots + r_m + b_2, b_1 + y + r_1 + \cdots + r_m + b_2) \in H \). Since \( N \) is zero-symmetric we must have that \( b_1 + y = -b_2 - r_m - \cdots - r_1 \). From the previous Lemma, \( -b_2 - r_m - \cdots - r_1 \in G_{n-1} \) which implies \( b_1 + y \in \bigcup Y_0 \cap G_{n-1} \), a contradiction. Hence \( \bigcup Y_0 \cap G_n = \emptyset \), \( \forall n \geq 0 \). If \( y \in \bigcup Y_0 \cup \bigcup X_n \) for some \( n \geq 1 \), then for some \( b_1 \in B_1 \), \( y + b_1 \in \bigcup Y_0 \cap G_n \) contradicting the previous situation. We have now shown that \( \bigcup Y_0 \cap X = \emptyset \). Suppose that \( \bigcup Y_m \cap X = \emptyset \), \( \forall n \geq 0 \). Let \( \mathbf{z} \in \bigcup Y_n \cap X \). Then for some \( b_1 \in B_1 \) and some \( k \geq 0 \), \( \mathbf{z} + b_1 \in \bigcup X_n \) and therefore \( \mathbf{z} + b_1 \in \bigcup Y_n \cap \bigcup X_n \). Let \( w \in \bigcup Y_n \) such that \( w \in \bigcup X_n \) and \( (z + b_1, w) \in H \). Since \( z + b_1 \in \bigcup X_n \), \( z + b_1 \in \bigcup X_{n+1} \). Thus \( w \in \bigcup X_{n+1} \cap \bigcup Y_n \), a contradiction. Consequently \( \bigcup Y_n \cap X = \emptyset \), \( \forall n \geq 0 \). If \( \mathbf{z} \in \bigcup Y_n \cap X \) for some \( n \geq 1 \), then \( \mathbf{z} = z_1 + b_2 \) for some \( z_1 \in \bigcup Y_{n-1} \), \( b_2 \in B_2 \) and \( z = z_2 + b_1 \) for some \( z_2 \in \bigcup X_n \), \( b_1 \in B_1 \). Since \( z_1 + b_2 = z_2 + b_1 \), \( z_1 - b_2 = z_2 - b_1 \). But \( z_1 - b_1 \in \bigcup Y_{n-1} \) and \( z_2 - b_2 \in \bigcup X_n \), a contradiction to \( \bigcup Y_{n-1} \cap X = \emptyset \). The result now follows.

**Lemma II.4.** If \( M(G, 2, H) \) is zero-symmetric, then (1) \( Y \cap (B_1 \oplus B_2) = \emptyset \).
(2) \( X \cap (B_1 \oplus B_2) = \emptyset \).

**Proof.** (1) We first show that \( \bigcup Y_n \cap (B_1 \oplus B_2) = \emptyset \), \( \forall n \geq 0 \). As in the proof of Lemma II.3 we can see that \( \bigcup Y_0 \cap (B_1 \oplus B_2) = \emptyset \). Suppose that \( \bigcup Y_m \cap (B_1 \oplus B_2) = \emptyset \), \( 0 \leq m < n \). If \( y = b_1 + b_2 \) for some \( y \in \bigcup Y_n \), \( b_1 \in B_1 \), \( b_2 \in B_2 \) then \( b_2 = y - b_1 \in \bigcup Y_n \cap B_2 \). If \( b_2 \neq 0 \) then \( \bigcup Y_n \cap \bigcup X_0 \neq \emptyset \), a contradiction while if \( b_2 = 0 \) then \( y = b_1 \in B_1 \) which implies that \( B_2 \in Y_n \) and therefore \( B_2 \cap \bigcup Y_{n-1} \neq \emptyset \), a contradiction to our assumption. Thus \( \bigcup Y_n \cap (B_1 \oplus B_2) = \emptyset \), \( \forall n \geq 0 \). If \( b_1 + b_2 \in \bigcup Y_n \), \( \forall n \geq 1 \), then for some \( b_2 \in B_2 \), \( b_1 + b_2 \neq \bigcup Y_n \), a contradiction. This establishes (1). The second statement can be shown in a similar way.

**Lemma II.5.** If \( M(G, 2, H) \) is zero-symmetric, then
(1) \( \bigcup Y_n \cap \bigcup_{k=0}^{n} Y_k = \emptyset \), \( \forall n \geq 0 \),
(2) \( \bigcup X_n \cap \bigcup_{k=0}^{n} X_k = \emptyset \), \( \forall n \geq 0 \).

**Proof.** (1) Let \( n \) be minimal so that \( \bigcup Y_n \cap \bigcup_{k=0}^{n} Y_k \neq \emptyset \). Obviously \( n \geq 1 \). Then \( \bigcup Y_m \cap \bigcup_{k=0}^{n} Y_k = \emptyset \) for all \( 0 \leq m < n \). Let \( y \in \bigcup Y_n \cap \bigcup_{k=0}^{n} Y_k \), say \( y \in \bigcup Y_j \) for some \( 1 \leq j \leq n \). Suppose that \( y \notin \bigcup Y_j \). Let \( x \in \bigcup Y_n \) such that \( x \in \bigcup Y_{n-1} \) and \( (y, x) \in H \). If \( j = 1 \) then \( x \in \bigcup Y_{j-1} \cap \bigcup Y_{j-1} \), a contradiction. Consequently \( y \notin \bigcup Y_{j-1} \), so \( y + b_2 \in \bigcup Y_{j-1} \) for some \( 0 \neq b_2 \in B_2 \). But then \( -(y + b_2) + y = -b_2 - y + y = -b_2 \in A_2 \). By Lemma II.4 \( -b_2 \in B_1^* \), a contradiction since \( B_1 \oplus B_2 = \{0\} \). The other statement follows similarly.

The finite case now follows from the above lemmas.
THEOREM II.6. Let $G$ be a finite group, $|G| \geq 3$ and $H = G/B_1 \times G/B_2$. If $M(G, 2, H)$ is zero-symmetric, then $B_1 = \{0\} = B_2$.

PROOF. Suppose that $B_1 \neq \{0\}$. Then $B_2 \neq \{0\}$ and since $G$ is finite there exists a positive integer $n$ such that $\bigcup \bigcup_{n=0}^{\infty} \bigcup_{\kappa=0}^{\infty} \bigcup_{\kappa} \neq \emptyset$. The result now follows from Lemma II.5.

From Theorem II.6 we note that when $G$ is a finite group and $H = G/B_1 \times G/B_2$ with $B_1 \neq \{0\}$ and $B_2 \neq \{0\}$ then there exists $x \in G^*$ such that $(x, x) \in H$. This implies that every group $H$ of this form contains a nontrivial subgroup of the diagonal $\{(x, x)|x \in G\}$. Therefore the associated near-ring $M(G, 2, H)$ cannot be zero-symmetric. We now present an example to show that this is not the case when $G$ is infinite.

EXAMPLE II.7. Let $\{0\} \neq A$ be a group with identity 0, $G = \bigoplus_{z \in Z} A = \{(x_z)_{z \in Z} \in A^Z|z = 0 \text{ for all but finitely many } z \in Z\}$. Further let $B_1 = \{(x_z)_{z \in Z} \in G|x_z = 0, \forall z \neq 0\}$ and $B_2 = \{(x_z)_{z \in Z} \in G|x_z = 0, \forall z \neq 1\}$. Define $\phi: G/B_1 \rightarrow G/B_2$ by $\phi((x_z)_{z \in Z} + B_1) = (y_z)_{z \in Z} + B_2$ where $y_z = x_{z-1}$ $\forall z \in Z$. It is straightforward to verify that $\phi$ is an isomorphism, so $\phi$ determines a 2-fold meromorphic product $H = G/B_1 \times G/B_2$ with $B_1 \neq \{0\}$ and $B_2 \neq \{0\}$. One can check that $(x_z)_{z \in Z} + B_1 \cap (x_z)_{z \in Z} + B_1 = \emptyset$ for all $(x_z)_{z \in Z} \in G \setminus B_1$. Since $B_1 \cap B_2 = \{0\}$ it follows that there is no $x \in G^*$ such that $(x, x) \in H$. Thus $M(G, 2, H)$ is zero-symmetric.

We are now ready to establish our major result.

THEOREM II.8. Let $G$ be an arbitrary group, $|G| \geq 3$ and $H = G/B_1 \times G/B_2$. If $N = M(G, 2, H)$ is a near-field, then $B_1 = \{0\} = B_2$.

PROOF. Case (A): We first suppose that $B_1 \neq \{0\}$ and $B_2 \neq \{0\}$. We may also assume that there is no $0 \neq x \in G$ with $(x, x) \in H$. We define a function $f: G \rightarrow G$ as follows.

(i) Let $f(0) = 0$, $f(b_{1\eta}) = b_{1\eta}$ for all $1 \leq \eta < \alpha$, $f(b_{2\eta}) = b_{2\eta}$ for all $1 \leq \gamma < \beta$ and if $b_{1\eta} + b_{2\gamma} \in B_1 \cap B_2$, $1 \leq \eta < \alpha$, $1 \leq \gamma < \beta$ let $f(b_{1\eta} + b_{2\gamma}) = b_{1\eta} + b_{2\eta}$. (ii) If $b_2 \in B_2$ define $f(x_1 + b_2) = x_1$. For $2 \leq \gamma < \beta$ let $f(x_\eta) = x_1$ and $f(x_\eta + b_2) = x_1 + b_{21}$ if $b_2 \in B_2$. Thus $f$ is defined on $\bigcup X_0$.

Suppose we have defined $f$ on $\bigcup X_m$ and $\bigcup X_m$ for each $0 \leq m < n$.

(iii) For $x \in \bigcup X_{n-1}$ and $b_1 \in B_1^*$ let $f(x + b_1) = b_{11} + f(x)$. This defines $f$ on $\bigcup X_n$.

(iv) In order to define $f$ on $\bigcup X_n$ we choose an arbitrary but fixed set of coset representatives $\{w_\xi|\xi < \delta_n\}$ of $\bigcup X_n$ such that for each $x_\xi + B_1 \in X_n$, $\sigma(x_\xi + B_1) = w_\xi + B_2$. If $\sigma(f(x_\xi) + B_1) = w_{\xi'} + B_2$ we define $f(w_\xi + b_2) = w_{\xi'} + b_{21}$ if $b_2 \in B_2^*$ and $f(w_\xi) = w_{\xi'}$.

In a similar manner we now define $f$ on $Y$.

(v) If $b_1 \in B_1$ let $f(y_1 + b_1) = y_1$. If $2 \leq \eta < \alpha$ let $f(y_\eta) = y_1$ and $f(y_\eta + b_1) = y_1 + b_{11}$ for $b_1 \in B_1^*$. Suppose $f$ has been defined on $\bigcup Y_m$ and $\bigcup Y_m$ for all $0 \leq m < n$.

(vi) For all $y \in \bigcup Y_{n-1}$ and $b_2 \in B_2^*$ let $f(y + b_2) = f(y) + b_{21}$.

(vii) Choose an arbitrary but fixed set of coset representatives $\{w_\xi|\xi < \delta_n\}$ of $\bigcup Y_n$ such that for $x_\xi + B_2 \in \bigcup Y_n$, $\sigma^{-1}(x_\xi + B_2) = w_\xi + B_1$. If $\sigma^{-1}(f(x_\xi) + B_2) = w_{\xi'} + B_1$ define $f(w_\xi) = w_{\xi'}$ and $f(w_\xi + b_1) = w_{\xi'} + b_{21}$ for $b_1 \in B_1^*$.

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We have now defined $f$ on $X \cup Y \cup B_1 \oplus B_2$.

(viii) Let $S = \{y+z|y \in \bigcup Y_n$ for some $n \geq 0, z \in \bigcup X_m$ for some $m \geq 0.\}$ For $y + x \in S$ let $f(y + x) = f(y) + f(x)$.

(ix) Finally define $f(z) = z$ if $z \notin X \cup Y \cup S \cup B_1 \oplus B_2$.

We now show that $f \in N$.

1: $f$ is well defined.

(i) Since $B_1 \cap B_2 = \{0\}$, $f$ is well defined on $B_1 \oplus B_2$.

(ii) Let $n \geq 1$. We need to show that $f$ is well defined on $X$.

Let $\{w_\xi|\xi < \delta\}$ be a set of representatives for the cosets in $X_{n-1}$ and let $y \in \bigcup X_n$. Then $y$ has the form $y = x + b_1$ for some $x \in \bigcup X_{n-1}, b_1 \in B_1$. Suppose $y = w_\xi + b_2 + b_1 = w_\xi' + b_2' + b_1'$, where $b_1, b_1', b_2, b_2', \xi \neq \xi'$. Then $-b_2' - w_\xi' + b_2 + b_1 = b_1' - b_1 \in G_{n-1} \cap B_1$. By Lemma II.4 and since $B_1 \cap B_2 = \{0\}$ this can only happen if $-b_2' - w_\xi' + b_2 + b_1 = 0$. But then $w_\xi + b_2 = w_\xi' + b_2'$, a contradiction. By Lemma II.5, $f$ is well defined on $X$.

(iii) Similar arguments show that $f$ is well defined on $Y$.

(iv) We show that $f$ is well defined on $S$. Suppose that $y_1 + x_1 = y_2 + x_2$, $y_1 \in \bigcup Y_j, y_2 \in \bigcup Y_i, x_1 \in \bigcup X_m, x_2 \in \bigcup X_n$. Then $-y_2 + y_1 = x_2 - x_1 \in Y \cup B_1 \cap X \cup B_2$ by Lemma II.2. According to Lemmas II.3 and II.4 this implies $y_1 = y_2$ and $x_1 = x_2$.

It is easy to show from Lemmas II.3 and II.4 that $S \cap X = \emptyset, S \cap Y = \emptyset$ and that $S \cap B_1 \oplus B_2 = \emptyset$. Suppose, for example, $y + x \in \bigcup X_\kappa \cap S$ for some $\kappa \geq 1$. Then $y + x = x' + b_1$ for some $x' \in \bigcup X_{\kappa-1}, b_1 \in B_1$. Then $y + x - b_1 = x'$, so $y + b_1' = x' - x \in Y \cap (X \cup B_2)$ for some $b_1' \in B_1$. This contradicts Lemmas II.3 and II.4.

Lemmas II.3 and II.4 now show that $f$ is well defined on $G$.

2: $f \in N$. Let $(x, y) \in H$. We must show that $(f(x), f(y)) \in H$.

Case (i) If $x \in B_1$, then $y \in B_2$, so $(f(x), f(y)) \in H$.

Case (ii) If $x = b_2 \gamma + b_1, 1 \leq \gamma < \beta, b_1 \in B_1$, then $y = x_\gamma + b_2$ for some $b_2 \in B_2$. Now $f(x) = b_2$ or $f(x) = b_{21} + b_{11}$, and $f(y) = x_1$ or $f(y) = x_1 + b_{21}$. In any case $(f(x), f(y)) \in H$.

Case (iii) Let $x \in X$. If $x \in \bigcup X_\kappa$ for some $\kappa \geq 1$, then $y \in \bigcup X_\kappa$. We have that $x \in x' + B_1$ for some $x' \in \bigcup X_{\kappa-1}$. By construction of $f$, $f(x) \in f(x') + B_1$ and $f(y) \in \sigma(f(x') + B_1)$. Thus $(f(x), f(y)) \in H$. If $x \in \bigcup X_\kappa$ for some $\kappa \geq 0$, then $x \in \bigcup X_{\kappa+1}$ and we are back to the previous case.

Case (iv) Let $x \in Y$. If $x = y_\eta + b_1 \in \bigcup Y_0$, then $y = b_{1\eta} + b_2, 1 \leq \eta < \alpha$ for some $b_2 \in B_2$. Now $f(x) = y_1$ or $f(x) = y_1 + b_{11}$ and $f(y) = b_{11}$ or $f(y) = b_{11} + b_{21}$. In any case $(f(x), f(y)) \in H$. Let $x \in \bigcup Y_n$ for some $n \geq 1$. Then $y \in \bigcup Y_n$ and with arguments similar to those in case (iii), we can show that $(f(x), f(y)) \in H$. Let $x \in \bigcup Y_1$. If $x \in \bigcup Y_0$ we have just seen that $(f(x), f(y)) \in H$. Hence we may assume that $x = x' + b_{2\gamma}$ for some $x' \in \bigcup Y_0, 1 \leq \gamma < \beta$. Let $1 \leq \eta < \alpha$ so that $(x', b_{1\eta}) \in H$. Hence $y = b_{1\eta} + x_\eta + b_2$ for some $b_2 \in B_2$. Now $f(x) = f(x' + b_{2\gamma}) = f(x') + b_{21}$ and $f(x') + b_{21} = y_1 + b_{11} + b_{21}$ or $f(x') + b_{21} = y_1 + b_{21}$. Further $f(y) = f(b_{1\eta} + x_\eta + b_2) = b_{11} + f(x_\eta + b_2)$ and $b_{11} + f(x_\eta + b_2) = b_{11} + x_1 + b_{21}$ or $b_{11} + f(x_\eta + b_2) = b_{11} + x_1$. In any case $(f(x), f(y)) \in H$.

Finally if $x \in \bigcup Y_n$ for some $n \geq 2$, then $x = x' + b_2$ for some $x' \in \bigcup Y_{n-1}, b_2 \in B_2$. We may assume that $b_2 = b_{2\gamma} \in B_2$. Let $y' \in \bigcup Y_{n-1}$ such that $y' \in \bigcup Y_{n-2}$ with $(x', y') \in H$. Then $(x' + b_{2\gamma}, y' + x_\gamma) \in H$, thus $y = y' + x_\gamma + b_2$.
for some \( b_2 \in B_2 \). Since \( y \in S \), \( (f(x), f(y)) = (f(x') + b_2 + f(y') + f(x + b_2)) \) which is either equal to \( (f(x') + b_2 + f(y') + f(x + b_2)) \) or equal to \( (f(x') + b_1 + f(y') + x + b_2) \). Since \( (f(x'), f(y')) \in H \) as shown previously and \( (b_2, x_1) \in H \) we have \( (f(x), f(y)) \in H \).

Case (v) Let \( x \in S \), say \( x = y^* + x^* \), \( y^* \in \bigcup Y_{n-1} \) and \( x^* \in \bigcup X_m \). Suppose that \( n \geq 1 \). Let \( \bar{y} \in \bigcup \bar{Y}_n \) such that \( \bar{y} \in \bigcup \bar{Y}_{n-1} \) with \( (y^*, \bar{y}) \in H \) and \( \bar{x} \in \bigcup \bar{X}_{m+1} \) such that \( (x^*, \bar{x}) \in H \). Then \( (y^* + x^*, \bar{y} + \bar{x}) \in H \). Thus \( y = \bar{y} + \bar{x} + b_2 \) for some \( b_2 \in B_2 \) and \( (f(x), f(y)) = (f(y^*) + f(x^*), f(\bar{y}) + f(\bar{x} + b_2)) \in H \) according to Cases (iii) and (iv). If \( n = 0 \), \( y^* \in \bigcup \bar{Y}_0 \). Let \( b_1^\eta \in B_1^\eta \) with \( (y^*, b_1^\eta) \in H \). Then \( (y^* + x^*, b_1^\eta + \bar{x}) \in H \). Thus \( y = b_1^\eta \bar{x} + \bar{x} + b_2 \) for some \( b_2 \in B_2 \). Hence \( (f(x), f(y)) = (f(y^*) + f(x^*), b_1^\eta + f(\bar{x} + b_2)) \in H \) since \( (f(x^*), f(\bar{x} + b_2)) \in H \).

Case (vi) Let \( x \in G \setminus (\bigcup Y \cup S \cup B_1 \oplus B_2) \). Then clearly \( y \notin B_1 \oplus B_2 \), \( y \notin \bigcup \bar{X}_\kappa \) for any \( \kappa \geq 0 \), \( y \notin Y \). Suppose \( y \in \bigcup X_{\kappa} \) for some \( \kappa \geq 1 \). If \( \kappa = 1 \), then \( y = b_1^\eta + \bar{y} \) for some \( \bar{y} \in \bigcup \bar{X}_0 \). Since \( y \notin \bigcup \bar{X}_0 \), \( b_1^\eta \neq 0 \). Now \( (b_2^\gamma, \bar{y}) \in H \) for some \( b_2^\gamma \in B_2^\gamma \). Hence \( (y^\eta + b_2^\gamma, b_1^\eta + \bar{y}) \in H \). Thus \( x = y^\eta + b_2^\gamma + b_1^\eta + b_2 \) for some \( b_1^\eta \in B_1 \). Hence \( x \in \bigcup Y_1 \), a contradiction. If \( \kappa \geq 2 \), \( y = b_1^\eta \bar{y} + \bar{y} \) for some \( \bar{y} \in \bigcup \bar{X}_{k-1} \), \( 1 \leq \eta < \alpha \). Let \( \bar{x} \in \bigcup \bar{X}_{k-1} \) such that \( (\bar{x}, \bar{y}) \in H \). Then \( x = y^\eta + \bar{x} + b_1^\eta \in S \), a contradiction. Finally let \( y \in S \). Then \( y = y^\eta + x_1 \) for some \( y^\eta \in \bigcup \bar{Y}_n \), \( x_1 \in \bigcup \bar{X}_m \). Suppose that \( m \geq 1 \). Let \( \bar{x}_1 \in \bigcup X_{m-1} \) such that \( \bar{x}_1 \in \bigcup \bar{X}_{m-1} \) with \( (\bar{x}_1, x_1) \in H \).

Since \( y^\eta \in \bigcup \bar{Y}_n \), \( y^\eta \in \bigcup Y_{n+1} \). Choose \( z \in \bigcup \bar{Y}_{n+1} \) with \( (z, y^\eta) \in H \). Then \( x = z + \bar{x}_1 + b_1 \) for some \( b_1 \in B_1 \). Hence \( x = z + b_1 + \bar{x}_1 \) for some \( b_1^\eta \in B_1 \). But then \( x \in S \), a contradiction. If \( m = 0 \) choose \( b_2^\gamma \in B_2^\gamma \) with \( (b_2^\gamma, x_1) \in H \). Then \( x = z + b_2^\gamma + b_1 + b_1 + b_2 \) for some \( b_1 \in B_1 \). Since \( x \in \bigcup \bar{Y}_{n+1}, x \in \bigcup \bar{Y}_{n+2} \), a contradiction. Therefore we must conclude that \( y \in G \setminus (\bigcup Y \cup S \cup B_1 \oplus B_2) \). Consequently \( (f(x), f(y)) = (x, y) \in H \).

It now follows that \( f \in N \). Clearly \( f \) is not invertible, since \( B_2 \neq \{0\} \) by assumption and \( f(x_1 + b_2) = x, \forall b_2 \in B_2 \).

Case (B). Suppose that either \( B_1 = \{0\} \) or \( B_2 = \{0\} \). W.l.o.g. we may assume that \( B_2 = \{0\}, B_1 \neq \{0\} \). Then \( X = \emptyset, S = \emptyset, B_1 \oplus B_2 = B_1 \). Define the function \( f : G \to G \) on \( Y \cup B_1 \) in the same way as before, noting that \( \bigcup \bar{Y}_n = \bigcup \bar{Y}_{n-1} \) for all \( n \geq 1 \). For \( z \notin Y \cup B_1 \) let \( f(z) = z \). As in Case (A) one can verify that \( f \) is well defined, \( f \in N \) but \( f \) is not invertible since \( f(y_1 + b_1) = y_1 \) for all \( b_1 \in B_1 \).

In both cases we obtain a noninvertible function \( f \in M(G, 2, H) \), a contradiction to \( N \) being a near-field. Thus \( B_1 = \{0\} = B_2 \).

We now show that the only near-fields arising in the case \( \kappa = 2 \) are fields. From the discussion prior to Lemma II.1 we see this is the case for \( |G| = 2 \). We now turn to \( |G| > 2 \). If \( A \) is a group of automorphisms of \( G \), then \( A^\circ \) denotes the group with the zero map adjoined.

**Theorem II.9.** Let \( |G| \geq 3 \). If \( N = M(G, 2, H) \) is a near-field then \( N \) is a field. In fact, \( N = M_A^\circ(G) \) where \( A = \langle \alpha \rangle \) is a cyclic group of automorphisms of \( G \) such that \( G = A_x \cup \{0\} \) for \( x \in G^* \). Conversely, if \( B = \langle \beta \rangle \) is a cyclic group of automorphisms of \( G \) such that \( G = B_x \cup \{0\} \) for \( x \in G^* \), then \( M_B^\circ(G) \) is a near-field and \( M_B^\circ(G) = M(G, 2, H) \) where \( H = G \times _\beta G \).

**Proof.** From Lemma II.1 and Theorem II.8, \( H = G \times _\alpha G = \{(x, \alpha(x)) | x \in G \} \).

But then \( f \in N \) if and only if \( \alpha f(x) = f(\alpha x) \) for all \( x \in G \), i.e., if and only if \( f \) belongs to the centralizer near-ring \( M(\alpha)^\circ(G) \). Since \( N \) is a near-field it is well

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known (see [1]) that $G^*$ must be the only nonzero orbit for $A = \langle \alpha \rangle$. Finally since $A$ is abelian, $M_{A^*}(G)$ is a field. For the converse it is clear that $M_{B^*}(G) = M(G, 2, H)$ with $H = G \times \beta G$ and again from [1] we see that $M(G, 2, H)$ is a near-field since $((\beta), G)$ satisfies the needed finiteness condition.

COROLLARY II.10. If $F$ is a finite field then $F \simeq M(G, 2, H)$ for some group $G$ and some 2-fold meromorphic product $H$.

PROOF. We know $F \simeq M_F(F)$ and $F^* = \langle \alpha \rangle$. By the above theorem, $F \simeq M(F, 2, F \times \alpha F)$.

COROLLARY II.11. Let $|G| \geq 3$. $H$ is an invariant linear 2-relation of the form $G \times \sigma G$ with $\langle \sigma \rangle$ transitive on $G^*$ if and only if $M(G, 2, H)$ is a field.

We conclude the paper with an example to show that the situation is quite different for $\kappa = 3$. Indeed, we see that one can have a meromorphic product $H = G / B_1 \times \sigma_1 G / B_2 \times \sigma_2 G / B_3$ such that $N = M(G, 3, H)$ is a near-field but $B_1 \neq \{0\}$, $B_2 \neq \{0\}$ and $B_3 \neq \{0\}$.

EXAMPLE II.12. Let $G = \mathbb{Z}_3^2$ with the usual basis $\{e_1, e_2, e_3, e_4\}$ and let $B_1 = B_2 = B_3 = G$. Let $\overline{B}_1 = \langle e_1 + e_2, e_3 + e_4 \rangle$, $\overline{B}_2 = \langle e_1, e_1 + e_3 + e_4 \rangle$ and $\overline{B}_3 = \langle e_1, e_1 + e_2 + e_3 \rangle$. The following scheme determines a meromorphic product $H$:

- $\overline{B}_1 \mapsto \overline{B}_2 \mapsto \overline{B}_3$,
- $e_1 + \overline{B}_1 \mapsto e_4 + \overline{B}_2 \mapsto e_1 + e_2 + e_4 + \overline{B}_3$,
- $e_1 + e_4 + \overline{B}_1 \mapsto e_1 + e_2 + \overline{B}_2 \mapsto e_1 + e_4 + \overline{B}_3$,
- $e_4 + \overline{B}_1 \mapsto e_2 + e_3 + \overline{B}_2 \mapsto e_2 + \overline{B}_3$.

One can check that $M(G, 3, H) = \{0, \text{id}\}$.

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DEPARTMENT OF MATHEMATICS, TEXAS A & M UNIVERSITY, COLLEGE STATION, TEXAS 77843 (Current address of C. J. Maxson)

Current address (Peter Fuchs): Department of Mathematics, Johannes Kepler Universität, A-4040 Linz, Austria