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NEAR-FIELDS ASSOCIATED WITH INVARIANT LINEAR κ-RELATIONS

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ABSTRACT. In this paper we investigate a construction method for subnear-rings of $M(G)$ proposed by H. Wielandt using subgroups of direct powers G^κ of G called invariant linear κ-relations. If $\kappa = 2$ we characterize, in terms of properties of these subgroups, when the associated near-rings are near-fields and prove that every near-field arising from an invariant linear 2-relation must be a field.

I. Introduction. In 1972 H. Wielandt [**7**] presented a very general method for constructing subnear-rings of the near-ring $M(G)$ of functions on the group G. A particular instance of this construction, namely centralizer near-rings, has been extensively investigated in the past several years. In this paper we initiate a study of the structure of the near-rings obtained by Wielandt's general method.

We recall the construction. Let $(G,+)$ be a group, let κ be a cardinal number and let G^κ denote the direct product of κ copies of G. We let $M(G)$ act on G^κ component-wise. For any subgroup H of G^κ we define

$$M(G,\kappa,H) = \{f \in M(G) | f(H) \subseteq H\}.$$

These $M(G,\kappa,H)$ are subnear-rings of $M(G)$ with identity id: $G \to G$, $\mathrm{id}(x) = x\ \forall x \in G$.

One is therefore led to an investigation of the transfer of information between the structure of the near-rings $M(G,\kappa,H)$ and the subgroups H of G^κ. Wielandt calls these subgroups invariant linear κ-relations and indicates that these linear κ-relations might be studied as in his work on permutation groups P via P-invariant κ-relations [**6**].

Another reason for investigating the near-rings $M(G,\kappa,H)$ is that they are indeed very general as indicated in the following theorem. Let R be a near-ring with identity, 1. It is well known that R can be embedded in $M(G)$ for some group G.

THEOREM I.1. *Let R be a near-ring with identity* 1. *Then there exists a group G, a cardinal number κ, and a subgroup H of G^κ such that $R \simeq M(G,\kappa,H)$.*

The reader is referred to the books by Meldrum [**2**] and Pilz [**3**] for the proof of this result as well as background information on near-rings.

In [**4**], Remak investigated the subgroup structure of G^2 and in [**5**] indicated how this can be extended to the case $\kappa \geq 3$. We briefly outline his results. Again let G be a group, κ a positive integer, $\kappa \geq 2$, and for $j \in \{1,\ldots,\kappa\}$ let B_j be a

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subgroup of G, $\overline{B}_j$ a normal subgroup of B_j such that $B_j/\overline{B}_j \simeq B_{j+1}/\overline{B}_{j+1}$ with isomorphisms σ_j, $j \in \{1, \ldots, \kappa - 1\}$. Let α be an ordinal, $\{b_{1\eta} | \eta < \alpha\}$ a set of coset representatives of $\overline{B}_1$ in B_1 where $b_{10} = 0$ and define a subset $H \subseteq G^\kappa$ by

$$H = \bigcup_{\eta<\alpha} \left[(b_{1\eta} + \overline{B}_1) \times \prod_{j=1}^{\kappa-1} (\sigma_j \circ \sigma_{j-1} \circ \cdots \circ \sigma_1 (b_{1\eta} + \overline{B}_1)) \right].$$

H is called a *κ-fold meromorphic product* and will be denoted by

$$H = B_1/\overline{B}_1 \times_{\sigma_1} B_2/\overline{B}_2 \times_{\sigma_2} \cdots \times_{\sigma_{\kappa-1}} B_\kappa/\overline{B}_\kappa.$$

It is straightforward to verify that H is a subgroup of G^κ but in general not every subgroup of G^κ is a κ-fold meromorphic product. For $\kappa = 2$, however we have such a result.

THEOREM I.2 (KLEIN-FRICKE) [4]. *Every subgroup of $G \times G$ is a 2-fold meromorphic product.*

In this paper we focus on near-fields for the case $\kappa = 2$. In the next section we characterize when $M(G, 2, H)$ is a near-field and find the somewhat surprising result that the only near-fields arising in this case are fields.

II. When is $M(G,2,H)$ a near-field? We now turn to a characterization of the triples $(G, 2, H)$ such that $M(G, 2, H)$ is a near-field. From the Klein-Fricke Theorem we know that $H = B_1/\overline{B}_1 \times_\sigma B_2/\overline{B}_2$. For $G = \mathbf{Z}_2$ the subgroups $H_1 = \mathbf{Z}_2/\mathbf{Z}_2 \times \{0\}/\{0\} = \mathbf{Z}_2 \times \{0\}$, $H_2 = \{0\} \times \mathbf{Z}_2$ and $H_3 = \{0\} \times \{0\}$ are such that $M(G, 2, H_i) \simeq \mathbf{Z}_2$, $i = 1, 2, 3$. For $H_4 = \{(0,0), (1,1)\}$ and $H_5 = \mathbf{Z}_2 \times \mathbf{Z}_2$ we get $M(G, 2, H_4) = M(G, 2, H_5) = M(\mathbf{Z}_2)$ which is not a near-field. For the remainder of the paper we take $|G| > 2$ and in a sequence of lemmas show that when $M(G, 2, H)$ is a near-field, H has the form $G \times_\sigma G$. For a subgroup S of G we let S^* denote $S \setminus \{0\}$.

LEMMA II.1. *Let $H = B_1/\overline{B}_1 \times_\sigma B_2/\overline{B}_2$. If $N = M(G, 2, H)$ is a near-field then $B_1 = B_2 = G$.*

PROOF. We may assume that $B_1 \cup B_2 \neq \{0\}$ since otherwise $M(G, 2, H) = M_0(G)$ which is not a near-field. If $B_1 \cup B_2 \neq G$ then the function $f \colon G \to G$ given by $f(x) = x$ if $x \in G \setminus (B_1 \cup B_2)$ and $f(x) = 0$ if $x \in B_1 \cup B_2$ is in N contradicting the fact that N is a near-field. Hence $B_1 \cup B_2 = G$ so at least one of B_1, B_2 must equal G, say $B_1 = G$. Suppose $B_2 \neq G$ and take $y \in G \setminus B_2$.

Case (i). $\overline{B}_1 \neq \{0\}$. Let $\hat{b}_1 \in \overline{B}_1^*$. One verifies that the function $h \colon G \to G$ defined by $h(y) = \hat{b}_1$ and $h(x) = 0$ for $x \neq y$ is in N. Since $|G| \geq 3$, h is not invertible, a contradiction.

Case (ii). $\overline{B}_1 = \{0\}$. Then $H = G/\{0\} \times_\sigma B_2/\overline{B}_2$. Now $\sigma(y) = b_2 + \overline{B}_2$ for some $b_2 \in B_2 \setminus \overline{B}_2$. Define $A_1 = \{x | x \in b_2 + \overline{B}_2\}$, $A_n = \bigcup \{\sigma(x) | x \in A_{n-1}\}$ for $n \geq 2$. Let $A = \bigcup_{n=1}^\infty A_n$. We define $f \colon G \to G$ by $f(x) = 0$, $x \in A \cup \{y\}$ and $f(x) = x$, $x \notin A \cup \{y\}$ and note that $f \in N$. If $0 \neq y' = y + b_2$, then $y' \notin A \cup \{y\}$ since $A \subseteq B_2$. Hence $f(y') = y'$ so f is not the zero map. Since f is not invertible, we have a contradiction.

Therefore we must conclude that $B_1 = B_2 = G$.

Now let $H = G/B_1 \times_\sigma G/B_2$ and let $N = M(G, 2, H)$. When N is a near-field, N is zero-symmetric since N contains the identity map and therefore cannot be isomorphic to the constant maps on $\mathbf{Z}_2$. Thus in this case we must have $B_1 \cap B_2 = \{0\}$, for if $0 \neq y \in B_1 \cap B_2$ then $f\colon G \to G$, $f(x) = y$ $\forall x \in G$ is an element of $N_c \backslash \{0\}$, a contradiction. Thus $B_1 \oplus B_2$ is a normal subgroup of G. We now develop some notation. Let α, β be ordinals such that $B_1 = \{b_{1\eta} | \eta < \alpha\}$, $B_2 = \{b_{2\gamma} | \gamma < \beta\}$ where $b_{10} = b_{20} = 0$. Further, if $B_1 \neq \{0\}$, let $\sigma(y_\eta + B_1) = b_{1\eta} + B_2$, $1 \leq \eta < \alpha$ and $\sigma(b_{2\gamma} + B_1) = x_\gamma + B_2$, $1 \leq \gamma < \beta$ if $B_2 \neq \{0\}$. Define $X_0 = \emptyset$, $\overline{X}_0 = \{x_\gamma + B_2 | 1 \leq \gamma < \beta\}$ and for $n \geq 1$, $X_n = \{x + B_1 | x + B_1 \cap w + B_2 \neq \emptyset$ for some $w + B_2 \in \overline{X}_{n-1}\}$, and $\overline{X}_n = \{\sigma(x + B_1) | x + B_1 \in X_n\}$. Then define $X = \bigcup_{n=0}^{\infty}(\bigcup X_n \cup \bigcup \overline{X}_n)$. If $B_2 = \{0\}$ define $X = \emptyset$. In a similar manner let $Y_0 = \emptyset$, $\overline{Y}_0 = \{y_\eta + B_1 | 1 \leq \eta < \alpha\}$, and for $n \geq 1$, $Y_n = \{y + B_2 | y + B_2 \cap w + B_1 \neq \emptyset$ for some $w + B_1 \in \overline{Y}_{n-1}\}$, $\overline{Y}_n = \{\sigma^{-1}(y + B_2) | y + B_2 \in Y_n\}$. Let $Y = \bigcup_{n=0}^{\infty}(\bigcup Y_n \cup \bigcup \overline{Y}_n)$. If $B_1 = \{0\}$ define $Y = \emptyset$. For Lemmas II.2–II.5 we always assume that $B_1 \neq \{0\}$ and $B_2 \neq \{0\}$. If either $B_1 = \{0\}$ or $B_2 = \{0\}$ then the results are trivial or can be seen to hold by making obvious modifications.

LEMMA II.2. *For $n \geq 0$ let $A_n = \bigcup_{\kappa=0}^{n} \bigcup \overline{Y}_\kappa \cup B_1$ and $G_n = \bigcup_{\kappa=0}^{n} \bigcup \overline{X}_\kappa \cup B_2$. Then A_n and G_n are subgroups of G for each $n \geq 0$.*

PROOF. We show that A_n is a subgroup of G for $n \geq 0$. A similar argument can be used for G_n, $n \geq 0$. Since $B_1 \oplus B_2 = B_2 \oplus B_1$ is a subgroup of G, so is $\bigcup \overline{Y}_0 \cup B_1$. Suppose we have shown that A_m is a subgroup for all $0 \leq m < n$. Let $z_1, z_2 \in A_n$.

Case (i). If $z_1, z_2 \in A_{n-1}$, then $z_1 - z_2 \in A_{n-1} \subseteq A_n$.

Case (ii). If $z_1, z_2 \in \bigcup \overline{Y}_n$ let $w_1, w_2 \in \bigcup Y_n$ such that $w_1, w_2 \in \bigcup \overline{Y}_{n-1}$ with $(z_1, w_1) \in H$, $(z_2, w_2) \in H$. Thus $(z_1 - z_2, w_1 - w_2) \in H$. If $w_1 - w_2 = 0$, then $z_1 - z_2 \in B_1$. If $w_1 - w_2 \in B_1^*$, then $z_1 - z_2 \in \bigcup \overline{Y}_0$. Finally, if $w_1 - w_2 \in \bigcup \overline{Y}_i$ for some $i \leq n-1$, then $w_1 - w_2 \in \bigcup Y_{i+1}$, thus $z_1 - z_2 \in \bigcup \overline{Y}_{i+1} \subseteq A_n$.

Case (iii). If $z_1 \in B_1$, $z_2 \in \bigcup \overline{Y}_n$, let $y \in \bigcup Y_n$ such that $y \in \bigcup \overline{Y}_{n-1}$ with $(z_2, y) \in H$. Since $(z_1, 0) \in H$, $(z_1 - z_2, -y) \in H$. Now $-y \in \bigcup \overline{Y}_i$ for some $i \leq n-1$, hence $-y \in \bigcup Y_{i+1}$ and $z_1 - z_2 \in \bigcup \overline{Y}_{i+1} \subseteq A_n$.

Case (iv). If $z_1 \in \bigcup \overline{Y}_n$, $z_2 \in \bigcup \overline{Y}_0$, let $w \in \bigcup \overline{Y}_{n-1}$ and $b_1 \in B_1^*$ such that $(z_1, w) \in H$, $(z_2, b_1) \in H$. Thus $(z_1 - z_2, w - b_1) \in H$. Since $w - b_1 \in \bigcup \overline{Y}_{n-1}$, $w - b_1 \in \bigcup Y_n$, thus $z_1 - z_2 \in \bigcup \overline{Y}_n \subseteq A_n$.

Case (v). Finally let $z_1 \in \bigcup \overline{Y}_i$ for some $n > i > 0$ and $z_2 \in \bigcup \overline{Y}_n$. Let $w_1 \in \bigcup \overline{Y}_{i-1}$, $w_2 \in \bigcup \overline{Y}_{n-1}$ such that $(z_1, w_1) \in H$, $(z_2, w_2) \in H$. Thus $(z_1 - z_2, w_1 - w_2) \in H$. Since $w_1 - w_2 \in A_{n-1}$, $w_1 - w_2 \in B_1$ or $w_1 - w_2 \in \bigcup \overline{Y}_i$ for some $i \leq n-1$. If $w_1 - w_2 = 0$ then $z_1 - z_2 \in B_1 \subseteq A_n$, if $w_1 - w_2 \in B_1^*$ then $z_1 - z_2 \in \bigcup \overline{Y}_0 \subseteq A_n$. If $w_1 - w_2 \in \bigcup \overline{Y}_i$ then $w_1 - w_2 \in \bigcup Y_{i+1}$, so $z_1 - z_2 \in \bigcup \overline{Y}_{i+1} \subseteq A_n$.

LEMMA II.3. *If $N = M(G, 2, H)$ is zero-symmetric then $Y \cap X = \emptyset$.*

PROOF. We first show that $\bigcup \overline{Y}_0 \cap X = \emptyset$. Suppose $y \in \bigcup \overline{Y}_0 \cap B_2$. Let $b_1 \in B_1^*$ such that $(y, b_1) \in H$. Since $y \in B_2$, $(b_1, y) \in H$. Thus $(y + b_1, b_1 + y) = (b_1 + y, b_1 + y) \in H$. Since N is zero-symmetric $b_1 + y = 0$, so $y = -b_1$. But $B_1 \cap B_2 = \{0\}$, hence $b_1 = 0$, a contradiction. Consequently $\bigcup \overline{Y}_0 \cap B_2 = \emptyset$. Suppose that $y \in \bigcup \overline{Y}_0 \cap \bigcup \overline{X}_0$. Let $b_1 \in B_1^*$, $b_2 \in B_2^*$ such that $(y, b_1) \in H$, $(b_2, y) \in H$. Then $(y + b_2, b_1 + y) \in H$ and $(y + b_2, b_1 + y) = (y + b_2, y + \bar{b}_1)$ for some $\bar{b}_1 \in B_1^*$. Since $(\bar{b}_1, b_2) \in H$, $(y + b_2 + \bar{b}_1, y + \bar{b}_1 + b_2) = (y + \bar{b}_1 + b_2, y + \bar{b}_1 + b_2) \in H$.

Thus $y+\bar{b}_1+b_2=0$ and $y+\bar{b}_1=-b_2$. Hence $y+\bar{b}_1\in\bigcup\overline{Y}_0\cap B_2$, a contradiction to our first statement. Therefore $\bigcup\overline{Y}_0\cap G_0=\varnothing$. Suppose that $\bigcup\overline{Y}_0\cap G_m=\varnothing$ for all $0\le m<n$ and $y\in\bigcup\overline{Y}_0\cap G_n$. Then $y\in\bigcup\overline{X}_n$ and there exists $r_1,\dots,r_m\in G_{n-1}$ and $b_2\in B_2^*$ such that $(r_1,y)\in H$, $(r_2,r_1)\in H,\dots(r_m,r_{m-1})\in H$ and $(b_2,r_m)\in H$. Therefore $(r_1+\cdots+r_m+b_2,y+r_1+\cdots+r_m)\in H$. Since $(y,b_1)\in H$ for some $b_1\in B_1^*$, $(y+r_1+\cdots+r_m+b_2,b_1+y+r_1+\cdots+r_m)\in H$. But $(b_1,0)\in H$ and $(0,b_2)\in H$ gives $(b_1+y+r_1+\cdots+r_m+b_2,b_1+y+r_1+\cdots+r_m+b_2)\in H$. Since N is zero-symmetric we must have that $b_1+y=-b_2-r_m-\cdots-r_1$. From the previous Lemma, $-b_2-r_m-\cdots-r_1\in G_{n-1}$ which implies $b_1+y\in\bigcup\overline{Y}_0\cap G_{n-1}$, a contradiction. Hence $\bigcup\overline{Y}_0\cap G_n=\varnothing$, $\forall n\ge 0$. If $y\in\bigcup\overline{Y}_0\cap\bigcup X_n$ for some $n\ge 1$, then for some $b_1\in B_1$, $y+b_1\in\bigcup\overline{Y}_0\cap G_{n-1}$ contradicting the previous situation. We have now shown that $\bigcup\overline{Y}_0\cap X=\varnothing$. Suppose that $\bigcup\overline{Y}_m\cap X=\varnothing$, $\forall 0\le m<n$. Let $z\in\bigcup\overline{Y}_n\cap X$. Then for some $b_1\in B_1$ and some $\kappa\ge 0$, $z+b_1\in\bigcup\overline{X}_\kappa$ and therefore $z+b_1\in\bigcup\overline{Y}_n\cap\bigcup\overline{X}_\kappa$. Let $w\in\bigcup Y_n$ such that $w\in\bigcup\overline{Y}_{n-1}$ and $(z+b_1,w)\in H$. Since $z+b_1\in\bigcup\overline{X}_\kappa$, $z+b_1\in\bigcup X_{\kappa+1}$. Thus $w\in\bigcup\overline{X}_{\kappa+1}\cap\bigcup\overline{Y}_{n-1}$, a contradiction. Consequently $\bigcup\overline{Y}_n\cap X=\varnothing$, $\forall n\ge 0$. If $z\in\bigcup Y_n\cap X$ for some $n\ge 1$, then $z=z_1+b_2$ for some $z_1\in\bigcup\overline{Y}_{n-1}$, $b_2\in B_2$ and $z=z_2+b_1$ for some $z_2\in\bigcup\overline{X}_\kappa$, $b_1\in B_1$. Since $z_1+b_2=z_2+b_1$, $z_1-b_1=z_2-b_2$. But $z_1-b_1\in\bigcup\overline{Y}_{n-1}$ and $z_2-b_2\in\bigcup\overline{X}_\kappa$, a contradiction to $\bigcup\overline{Y}_{n-1}\cap X=\varnothing$. The result now follows.

LEMMA II.4. *If $M(G,2,H)$ is zero-symmetric, then* (1) $Y\cap(B_1\oplus B_2)=\varnothing$. (2) $X\cap(B_1\oplus B_2)=\varnothing$.

PROOF. (1) We first show that $\bigcup\overline{Y}_n\cap(B_1\oplus B_2)=\varnothing$, $\forall n\ge 0$. As in the proof of Lemma II.3 we can see that $\bigcup\overline{Y}_0\cap(B_1\oplus B_2)=\varnothing$. Suppose that $\bigcup\overline{Y}_m\cap(B_1\oplus B_2)=\varnothing$, $0\le m<n$. If $y=b_1+b_2$ for some $y\in\bigcup\overline{Y}_n$, $b_1\in B_1$, $b_2\in B_2$ then $b_2=y-b_1\in\bigcup\overline{Y}_n\cap B_2$. If $b_2\ne 0$ then $\bigcup Y_n\cap\bigcup\overline{X}_0\ne\varnothing$, a contradiction while if $b_2=0$ then $y=b_1\in B_1$ which implies that $B_2\in Y_n$ and therefore $B_2\cap\bigcup\overline{Y}_{n-1}\ne\varnothing$, a contradiction to our assumption. Thus $\bigcup\overline{Y}_n\cap(B_1\oplus B_2)=\varnothing$, $\forall n\ge 0$. If $b_1+b_2\in\bigcup Y_n$ for some $n\ge 1$, then for some $\bar{b}_2\in B_2$, $b_1+b_2+\bar{b}_2\in\bigcup\overline{Y}_{n-1}$, a contradiction. This establishes (1). The second statement can be shown in a similar way.

LEMMA II.5. *If $M(G,2,H)$ is zero-symmetric, then*
(1) $\bigcup\overline{Y}_n\cap\bigcup_{\kappa=0}^n\bigcup Y_\kappa=\varnothing$, $\forall n\ge 0$,
(2) $\bigcup\overline{X}_n\cap\bigcup_{\kappa=0}^n\bigcup X_\kappa=\varnothing$, $\forall n\ge 0$.

PROOF. (1) Let n be minimal so that $\bigcup\overline{Y}_n\cap\bigcup_{\kappa=0}^n\bigcup Y_\kappa\ne\varnothing$. Obviously $n\ge 1$. Then $\bigcup\overline{Y}_m\cap\bigcup_{\kappa=0}^m\bigcup Y_\kappa=\varnothing$ for all $0\le m<n$. Let $y\in\bigcup\overline{Y}_n\cap\bigcup_{\kappa=0}^n\bigcup Y_\kappa$, say $y\in\bigcup Y_j$ for some $1\le j\le n$. Suppose that $y\in\bigcup\overline{Y}_{j-1}$. Let $x\in\bigcup Y_n$ such that $x\in\bigcup\overline{Y}_{n-1}$ and $(y,x)\in H$. If $j-1\ne 0$ then $x\in\bigcup\overline{Y}_{n-1}\cap\bigcup Y_{j-1}$, a contradiction. If $j-1=0$, then $x\in B_1\oplus B_2\cap\bigcup\overline{Y}_{n-1}$ which contradicts Lemma II.4. Consequently $y\notin\bigcup\overline{Y}_{j-1}$, so $y+b_2\in\bigcup\overline{Y}_{j-1}$ for some $0\ne b_2\in B_2$. But then $-(y+b_2)+y=-b_2-y+y=-b_2\in A_n$. By Lemma II.4 $-b_2\in B_1^*$, a contradiction since $B_1\cap B_2=\{0\}$. The other statement follows similarly.

The finite case now follows from the above lemmas.

THEOREM II.6. *Let G be a finite group, $|G| \geq 3$ and $H = G/B_1 \times_\sigma G/B_2$. If $M(G, 2, H)$ is zero-symmetric, then $B_1 = \{0\} = B_2$.*

PROOF. Suppose that $B_1 \neq \{0\}$. Then $B_2 \neq \{0\}$ and since G is finite there exists a positive integer n such that $\bigcup \overline{Y}_n \cap \bigcup_{\kappa=0}^{n} \bigcup Y_\kappa \neq \varnothing$. The result now follows from Lemma II.5.

From Theorem II.6 we note that when G is a finite group and $H = G/B_1 \times_\sigma G/B_2$ with $B_1 \neq \{0\}$ and $B_2 \neq \{0\}$ then there exists $x \in G^*$ such that $(x, x) \in H$. This implies that every group H of this form contains a nontrivial subgroup of the diagonal $\{(x, x) | x \in G\}$. Therefore the associated near-ring $M(G, 2, H)$ cannot be zero-symmetric. We now present an example to show that this is *not* the case when G is infinite.

EXAMPLE II.7. Let $\{0\} \neq A$ be a group with identity 0, $G = \bigoplus_{z \in \mathbf{Z}} A = \{(x_z)_{z \in \mathbf{Z}} \in A^{\mathbf{Z}} | x_z = 0$ for all but finitely many $z \in \mathbf{Z}\}$. Further let $B_1 = \{(x_z)_{z \in \mathbf{Z}} \in G | x_z = 0, \forall z \neq 0\}$ and $B_2 = \{(x_z)_{z \in \mathbf{Z}} \in G | x_z = 0, \forall z \neq 1\}$. Define $\phi\colon G/B_1 \to G/B_2$ by $\phi((x_z)_{z \in \mathbf{Z}} + B_1) = (y_z)_{z \in \mathbf{Z}} + B_2$ where $y_z = x_{z-1} \forall z \in \mathbf{Z}$. It is straightforward to verify that ϕ is an isomorphism, so ϕ determines a 2-fold meromorphic product $H = G/B_1 \times_\phi G/B_2$ with $B_1 \neq \{0\}$ and $B_2 \neq \{0\}$. One can check that $(x_z)_{z \in \mathbf{Z}} + B_1 \cap \phi((x_z)_{z \in \mathbf{Z}} + B_1) = \varnothing$ for all $(x_z)_{z \in \mathbf{Z}} \in G \setminus B_1$. Since $B_1 \cap B_2 = \{0\}$ it follows that there is no $x \in G^*$ such that $(x, x) \in H$. Thus $M(G, 2, H)$ is zero-symmetric.

We are now ready to establish our major result.

THEOREM II.8. *Let G be an arbitrary group, $|G| \geq 3$ and $H = G/B_1 \times_\sigma G/B_2$. If $N = M(G, 2, H)$ is a near-field, then $B_1 = \{0\} = B_2$.*

PROOF. Case (A): We first suppose that $B_1 \neq \{0\}$ and $B_2 \neq \{0\}$. We may also assume that there is no $0 \neq x \in G$ with $(x, x) \in H$. We define a function $f\colon G \to G$ as follows.

(i) Let $f(0) = 0$, $f(b_{1\eta}) = b_{11}$ for all $1 \leq \eta < \alpha$, $f(b_{2\gamma}) = b_{21}$ for all $1 \leq \gamma < \beta$ and if $b_{1\eta} + b_{2\gamma} \in B_1 \oplus B_2$, $1 \leq \eta < \alpha$, $1 \leq \gamma < \beta$ let $f(b_{1\eta} + b_{2\gamma}) = b_{11} + b_{21}$.

(ii) If $b_2 \in B_2$ define $f(x_1 + b_2) = x_1$. For $2 \leq \gamma < \beta$ let $f(x_\gamma) = x_1$ and $f(x_\gamma + b_2) = x_1 + b_{21}$ if $b_2 \in B_2^*$. Thus f is defined on $\bigcup \overline{X}_0$.

Suppose we have defined f on $\bigcup X_m$ and $\bigcup \overline{X}_m$ for each $0 \leq m < n$.

(iii) For $x \in \bigcup \overline{X}_{n-1}$ and $b_1 \in B_1^*$ let $f(x + b_1) = b_{11} + f(x)$. This defines f on $\bigcup X_n$.

(iv) In order to define f on $\bigcup \overline{X}_n$ we choose an arbitrary but fixed set of coset representatives $\{w_\xi | \xi < \delta_n\}$ of $\overline{X}_n$ such that for each $z_\xi + B_1 \in X_n$, $\sigma(z_\xi + B_1) = w_\xi + B_2$. If $\sigma(f(z_\xi) + B_1) = w_{\xi'} + B_2$ we define $f(w_\xi + b_2) = w_{\xi'} + b_{21}$ if $b_2 \in B_2^*$ and $f(w_\xi) = w_{\xi'}$.

In a similar manner we now define f on Y.

(v) If $b_1 \in B_1$ let $f(y_1 + b_1) = y_1$. If $2 \leq \eta < \alpha$ let $f(y_\eta) = y_1$ and $f(y_\eta + b_1) = y_1 + b_{11}$ for $b_1 \in B_1^*$.

Suppose f has been defined on $\bigcup Y_m$ and $\bigcup \overline{Y}_m$ for all $0 \leq m < n$.

(vi) For all $y \in \bigcup \overline{Y}_{n-1}$ and $b_2 \in B_2^*$ let $f(y + b_2) = f(y) + b_{21}$.

(vii) Choose an arbitrary but fixed set of coset representatives $\{w_\xi | \xi < \delta_n\}$ of $\overline{Y}_n$ such that for $z_\xi + B_2 \in \bigcup Y_n$, $\sigma^{-1}(z_\xi + B_2) = w_\xi + B_1$. If $\sigma^{-1}(f(z_\xi) + B_2) = w_{\xi'} + B_1$ define $f(w_\xi) = w_{\xi'}$ and $f(w_\xi + b_1) = w_{\xi'} + b_{11}$ for $b_1 \in B_1^*$.

We have now defined f on $X \cup Y \cup B_1 \oplus B_2$.

(viii) Let $S = \{y + x | y \in \bigcup \overline{Y}_n$ for some $n \geq 0$, $x \in \bigcup \overline{X}_m$ for some $m \geq 0.\}$ For $y + x \in S$ let $f(y + x) = f(y) + f(x)$.

(ix) Finally define $f(z) = z$ if $z \notin X \cup Y \cup S \cup B_1 \oplus B_2$.

We now show that $f \in N$.

1: f is well defined.

(i) Since $B_1 \cap B_2 = \{0\}$, f is well defined on $B_1 \oplus B_2$.

(ii) Let $n \geq 1$. We need to show that f is well defined on $\bigcup X_n$.

Let $\{w_\xi | \xi < \delta\}$ be a set of representatives for the cosets in $\overline{X}_{n-1}$ and let $y \in \bigcup X_n$. Then y has the form $y = x + b_1$ for some $x \in \bigcup \overline{X}_{n-1}$, $b_1 \in B_1$. Suppose $y = w_\xi + b_2 + b_1 = w_{\xi'} + b_2' + b_1'$, where $b_1, b_1' \in B_1$, $b_2, b_2' \in B_2$, $\xi \neq \xi'$. Then $-b_2' - w_{\xi'} + w_\xi + b_2 = b_1' - b_1 \in G_{n-1} \cap B_1$. By Lemma II.4 and since $B_1 \cap B_2 = \{0\}$ this can only happen if $-b_2' - w_{\xi'} + w_\xi + b_2 = 0$. But then $w_\xi + b_2 = w_{\xi'} + b_2'$, a contradiction. By Lemma II.5, f is well defined on X.

(iii) Similar arguments show that f is well defined on Y.

(iv) We show that f is well defined on S. Suppose that $y_1 + x_1 = y_2 + x_2$, $y_1 \in \bigcup \overline{Y}_j$, $y_2 \in \bigcup \overline{Y}_i$, $x_1 \in \bigcup \overline{X}_m$, $x_2 \in \bigcup \overline{X}_n$. Then $-y_2 + y_1 = x_2 - x_1 \in Y \cup B_1 \cap X \cup B_2$ by Lemma II.2. According to Lemmas II.3 and II.4 this implies $y_1 = y_2$ and $x_1 = x_2$.

It is easy to show from Lemmas II.3 and II.4 that $S \cap X = \varnothing$, $S \cap Y = \varnothing$ and that $S \cap B_1 \oplus B_2 = \varnothing$. Suppose, for example, $y + x \in \bigcup X_\kappa \cap S$ for some $\kappa \geq 1$. Then $y + x = x' + b_1$ for some $x' \in \bigcup \overline{X}_{\kappa-1}$, $b_1 \in B_1$. Then $y + x - b_1 = x'$, so $y + b_1' = x' - x \in Y \cap (X \cup B_2)$ for some $b_1' \in B_1$. This contradicts Lemmas II.3 and II.4.

Lemmas II.3 and II.4 now show that f is well defined on G.

2: $f \in N$. Let $(x, y) \in H$. We must show that $f((x, y)) = (f(x), f(y)) \in H$.

Case (i) If $x \in B_1$, then $y \in B_2$, so $(f(x), f(y)) \in H$.

Case (ii) If $x = b_{2\gamma} + b_1$, $1 \leq \gamma < \beta$, $b_1 \in B_1$, then $y = x_\gamma + b_2$ for some $b_2 \in B_2$. Now $f(x) = b_{21}$ or $f(x) = b_{21} + b_{11}$, and $f(y) = x_1$ or $f(y) = x_1 + b_{21}$. In any case $(f(x), f(y)) \in H$.

Case (iii) Let $x \in X$. If $x \in \bigcup X_\kappa$ for some $\kappa \geq 1$, then $y \in \bigcup \overline{X}_\kappa$. We have that $x \in x' + B_1$ for some $x' \in \bigcup \overline{X}_{\kappa-1}$. By construction of f, $f(x) \in f(x') + B_1$ and $f(y) \in \sigma(f(x') + B_1)$. Thus $(f(x), f(y)) \in H$. If $x \in \bigcup \overline{X}_\kappa$ for some $\kappa \geq 0$, then $x \in \bigcup X_{\kappa+1}$ and we are back to the previous case.

Case (iv) Let $x \in Y$. If $x = y_\eta + b_1 \in \bigcup \overline{Y}_0$, then $y = b_{1\eta} + b_2$, $1 \leq \eta < \alpha$ for some $b_2 \in B_2$. Now $f(x) = y_1$ or $f(x) = y_1 + b_{11}$ and $f(y) = b_{11}$ or $f(y) = b_{11} + b_{21}$. In any case $(f(x), f(y)) \in H$. Let $x \in \bigcup \overline{Y}_n$ for some $n \geq 1$. Then $y \in \bigcup Y_n$ and with arguments similar to those in case (iii), we can show that $(f(x), f(y)) \in H$. Let $x \in \bigcup Y_1$. If $x \in \bigcup \overline{Y}_0$ we have just seen that $(f(x), f(y)) \in H$. Hence we may assume that $x = x' + b_{2\gamma}$ for some $x' \in \bigcup \overline{Y}_0$, $1 \leq \gamma < \beta$. Let $1 \leq \eta < \alpha$ so that $(x', b_{1\eta}) \in H$. Hence $y = b_{1\eta} + x_\gamma + b_2$ for some $b_2 \in B_2$. Now $f(x) = f(x' + b_{2\gamma}) = f(x') + b_{21}$ and $f(x') + b_{21} = y_1 + b_{11} + b_{21}$ or $f(x') + b_{21} = y_1 + b_{21}$. Further $f(y) = f(b_{1\eta} + x_\gamma + b_2) = b_{11} + f(x_\gamma + b_2)$ and $b_{11} + f(x_\gamma + b_2) = b_{11} + x_1 + b_{21}$ or $b_{11} + f(x_\gamma + b_2) = b_{11} + x_1$. In any case $(f(x), f(y)) \in H$.

Finally if $x \in \bigcup Y_n$ for some $n \geq 2$, then $x = x' + b_2$ for some $x' \in \bigcup \overline{Y}_{n-1}$, $b_2 \in B_2$. We may assume that $b_2 = b_{2\gamma} \in B_2^*$. Let $y' \in \bigcup Y_{n-1}$ such that $y' \in \bigcup \overline{Y}_{n-2}$ with $(x', y') \in H$. Then $(x' + b_{2\gamma}, y' + x_\gamma) \in H$, thus $y = y' + x_\gamma + b_2$

for some $b_2 \in B_2$. Since $y \in S$, $(f(x), f(y)) = (f(x')+b_{21}, f(y')+f(x_\gamma+b_2))$ which is either equal to $(f(x')+b_{21}, f(y')+x_1+b_{21})$ or equal to $(f(x')+b_{21}, f(y')+x_1)$. Since $(f(x'), f(y')) \in H$ as shown previously and $(b_{21}, x_1) \in H$ we have $(f(x), f(y)) \in H$.

Case (v) Let $x \in S$, say $x = y^* + x^*$, $y^* \in \bigcup \overline{Y}_n$, $x^* \in \bigcup \overline{X}_m$. Suppose that $n \geq 1$. Let $\overline{y} \in \bigcup Y_n$ such that $\overline{y} \in \bigcup \overline{Y}_{n-1}$ with $(y^*, \overline{y}) \in H$ and $\overline{x} \in \bigcup \overline{X}_{m+1}$ such that $(x^*, \overline{x}) \in H$. Then $(y^* + x^*, \overline{y} + \overline{x}) \in H$. Thus $y = \overline{y} + \overline{x} + b_2$ for some $b_2 \in B_2$ and $(f(x), f(y)) = (f(y^*) + f(x^*), f(\overline{y}) + f(\overline{x} + b_2)) \in H$ according to Cases (iii) and (iv). If $n = 0$, $y^* \in \bigcup \overline{Y}_0$. Let $b_{1\eta} \in B_1^*$ with $(y^*, b_{1\eta}) \in H$. Then $(y^* + x^*, b_{1\eta} + \overline{x}) \in H$. Thus $y = b_{1\eta} + \overline{x} + b_2$ for some $b_2 \in B_2$. Hence $(f(x), f(y)) = (f(y^*) + f(x^*), b_{11} + f(\overline{x} + b_2)) \in H$, since $(f(x^*), f(\overline{x} + b_2)) \in H$.

Case (vi) Let $x \in G \setminus (X \cup Y \cup S \cup B_1 \oplus B_2)$. Then clearly $y \notin B_1 \oplus B_2$, $y \notin \bigcup \overline{X}_\kappa$ for any $\kappa \geq 0$, $y \notin Y$. Suppose $y \in \bigcup X_\kappa$ for some $\kappa \geq 1$. If $\kappa = 1$, then $y = b_1 + \overline{y}$ for some $\overline{y} \in \bigcup \overline{X}_0$. Since $y \notin \bigcup \overline{X}_0$, $b_1 = b_{1\eta} \neq 0$. Now $(b_{2\gamma}, \overline{y}) \in H$ for some $b_{2\gamma} \in B_2^*$. Hence $(y_\eta + b_{2\gamma}, b_{1\eta} + \overline{y}) \in H$. Thus $x = y_\eta + b_{2\gamma} + b_1$ for some $b_1 \in B_1$. Hence $x \in \bigcup Y_1$, a contradiction. If $\kappa \geq 2$ $y = b_{1\eta} + \overline{y}$ for some $\overline{y} \in \bigcup \overline{X}_{\kappa-1}$, $1 \leq \eta < \alpha$. Let $\overline{x} \in \bigcup \overline{X}_{\kappa-2}$ such that $(\overline{x}, \overline{y}) \in H$. Then $x = y_\eta + \overline{x} + b_1 \in S$, a contradiction. Finally let $y \in S$. Then $y = y_1 + x_1$ for some $y_1 \in \bigcup \overline{Y}_n$, $x_1 \in \bigcup \overline{X}_m$. Suppose that $m \geq 1$. Let $\overline{x}_1 \in \bigcup X_m$ such that $\overline{x}_1 \in \bigcup \overline{X}_{m-1}$ with $(\overline{x}_1, x_1) \in H$.

Since $y_1 \in \bigcup \overline{Y}_n$, $y_1 \in \bigcup Y_{n+1}$. Choose $z \in \bigcup \overline{Y}_{n+1}$ with $(z, y_1) \in H$. Then $(z + \overline{x}_1, y_1 + x_1) \in H$. Thus $x = z + \overline{x}_1 + b_1$ for some $b_1 \in B_1$. Hence $x = z + b_1^* + \overline{x}_1$ for some $b_1^* \in B_1$. But then $x \in S$, a contradiction. If $m = 0$ choose $b_{2\gamma} \in B_2^*$ with $(b_{2\gamma}, x_1) \in H$. Then $(z + b_{2\gamma}, y_1 + x_1) \in H$. Thus $x = z + b_{2\gamma} + b_1 = z + b_1 + b_{2\gamma}$ for some $b_1 \in B_1$. Since $z + b_1 \in \bigcup \overline{Y}_{n+1}$, $x \in \bigcup Y_{n+2}$, a contradiction. Therefore we must conclude that $y \in G \setminus (X \cup Y \cup S \cup B_1 \oplus B_2)$. Consequently $(f(x), f(y)) = (x, y) \in H$.

It now follows that $f \in N$. Clearly f is not invertible, since $B_2 \neq \{0\}$ by assumption and $f(x_1 + b_2) = x$, $\forall b_2 \in B_2$.

Case (B). Suppose that either $B_1 = \{0\}$ or $B_2 = \{0\}$. W.l.o.g. we may assume that $B_2 = \{0\}$, $B_1 \neq \{0\}$. Then $X = \varnothing$, $S = \varnothing$, $B_1 \oplus B_2 = B_1$. Define the function $f: G \to G$ on $Y \cup B_1$ in the same way as before, noting that $\bigcup Y_n = \bigcup \overline{Y}_{n-1}$ for all $n \geq 1$. For $z \notin Y \cup B_1$ let $f(z) = z$. As in Case (A) one can verify that f is well defined, $f \in N$ but f is not invertible since $f(y_1 + b_1) = y_1$ for all $b_1 \in B_1$.

In both cases we obtain a noninvertible function $f \in M(G, 2, H)$, a contradiction to N being a near-field. Thus $B_1 = \{0\} = B_2$.

We now show that the only near-fields arising in the case $\kappa = 2$ are fields. From the discussion prior to Lemma II.1 we see this is the case for $|G| = 2$. We now turn to $|G| > 2$. If A is a group of automorphisms of G, then A° denotes the group with the zero map adjoined.

THEOREM II.9. *Let* $|G| \geq 3$. *If* $N = M(G, 2, H)$ *is a near-field then* **N** *is a field. In fact,* $N = M_{A^\circ}(G)$ *where* $A = \langle \alpha \rangle$ *is a cyclic group of automorphisms of* G *such that* $G = Ax \cup \{0\}$ *for* $x \in G^*$. *Conversely, if* $B = \langle \beta \rangle$ *is a cyclic group of automorphisms of* G *such that* $G = Bx \cup \{0\}$ *for* $x \in G^*$, *then* $M_{B^\circ}(G)$ *is a near-field and* $M_{B^\circ}(G) = M(G, 2, H)$ *where* $H = G \times_\beta G$.

PROOF. From Lemma II.1 and Theorem II.8, $H = G \times_\alpha G = \{(x, \alpha(x)) | x \in G\}$. But then $f \in N$ if and only if $\alpha f(x) = f(\alpha x)$ for all $x \in G$, i.e., if and only if f belongs to the centralizer near-ring $M_{\langle \alpha \rangle^\circ}(G)$. Since N is a near-field it is well known (see [1]) that G^* must be the only nonzero orbit for $A = \langle\alpha\rangle$. Finally since A is abelian, $M_{A^\circ}(G)$ is a field. For the converse it is clear that $M_{B^\circ}(G) = M(G,2,H)$ with $H = G\times_\beta G$ and again from [1] we see that $M(G,2,H)$ is a near-field since $(\langle\beta\rangle, G)$ satisfies the needed finiteness condition.

COROLLARY II.10. *If F is a finite field then $F \simeq M(G,2,H)$ for some group G and some 2-fold meromorphic product H.*

PROOF. We know $F \simeq M_F(F)$ and $F^* = \langle\alpha\rangle$. By the above theorem, $F \simeq M(F,2,F\times_\alpha F)$.

COROLLARY II.11. *Let $|G| \geq 3$. H is an invariant linear 2-relation of the form $G\times_\sigma G$ with $\langle\sigma\rangle$ transitive on G^* if and only if $M(G,2,H)$ is a field.*

We conclude the paper with an example to show that the situation is quite different for $\kappa = 3$. Indeed, we see that one can have a meromorphic product $H = G/B_1\times_{\sigma_1}G/B_2\times_{\sigma_2}G/B_3$ such that $N = M(G,3,H)$ is a near-field but $B_1 \neq \{0\}$, $B_2 \neq \{0\}$ and $B_3 \neq \{0\}$.

EXAMPLE II.12. Let $G = \mathbf{Z}_2^4$ with the usual basis $\{e_1,e_2,e_3,e_4\}$ and let $B_1 = B_2 = B_3 = G$. Let $\overline{B}_1 = \langle e_1+e_2, e_3+e_4\rangle$, $\overline{B}_2 = \langle e_1, e_1+e_3+e_4\rangle$ and $\overline{B}_3 = \langle e_1, e_1+e_2+e_3\rangle$. The following scheme determines a meromorphic product H:

$$\begin{gathered}
\overline{B}_1 \mapsto \overline{B}_2 \mapsto \overline{B}_3, \\
e_1 + \overline{B}_1 \mapsto e_4 + \overline{B}_2 \mapsto e_1 + e_2 + e_4 + \overline{B}_3, \\
e_1 + e_4 + \overline{B}_1 \mapsto e_1 + e_2 + \overline{B}_2 \mapsto e_1 + e_4 + \overline{B}_3, \\
e_4 + \overline{B}_1 \mapsto e_2 + e_3 + \overline{B}_2 \mapsto e_2 + \overline{B}_3.
\end{gathered}$$

One can check that $M(G,3,H) = \{0, \mathrm{id}\}$.

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