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PLANE CURVES WHOSE SINGULAR POINTS ARE CUSPS

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ABSTRACT. Let C be an irreducible curve of degree d in the complex projective plane. We assume that each singular point is a one place point with multiplicity 2 or 3. Let σ be the sum of "the Milnor numbers" of the singularities. Then we shall show that $7\sigma < 6d^2 - 9d$. This gives a necessary condition for the existence of such a curve, for example, if C is rational, then $d \leq 10$.

1. Introduction. Let C be an irreducible curve of degree d in the complex projective plane $\mathbf{P}^2$. We assume that C is not smooth and each singular point is a cusp (i.e., one place point) with multiplicity 2 or 3. Let P_i be the singular point and μ_i be the Milnor number of the singularity at P_i, where $i = 1, 2, \ldots$. Then, putting $\sigma = \sum_i 6[\mu_i/6]$, where [ ] denotes the Gauss' symbol, we have the following inequality (cf. [**1**, §5]).

THEOREM 1. *Let C be the above-mentioned curve. Then $7\sigma < 6d^2 - 9d$.*

There has been a problem whether there exist curves in $\mathbf{P}^2$ with assigned numerical characters satisfying the genus formula of Clebsch [**3**, §9.1]. More than half a century ago Lefschetz and Zariski studied such a problem for Plückerian characters [**4**, **9**]. Now, let g be the genus of the normalization of C. Then from the above theorem we obtain the following inequality.

COROLLARY 2. $14g \geq d^2 - 12d + 16$. *Especially, if C is rational, then $3 \leq d \leq 10$.*

REMARK 3. The genus formula implies $\limsup_{d\to\infty} \sigma/d^2 \leq 1$, but the above theorem gives a better inequality $\limsup_{d\to\infty} \sigma/d^2 \leq 6/7$ (cf. [**2**, §8]).

2. Miyaoka's inequality. The key to the proof of the theorem is a result of Miyaoka [**5**, Corollary 1.2]. Before stating it we list the necessary notations.

X: a projective nonsingular surface,

D: a reduced divisor on X with normal crossings,

K: the canonical divisor on X,

$e(V)$: the topological Euler characteristic of a variety V. Then the result is stated as follows.

LEMMA 1. *If $K + D$ is numerically equivalent to an effective rational divisor, then $3\{e(X) - e(D)\} \geq (K + D)^2$. If the equality holds, then D is a semistable curve.*

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The outline of the proof of Theorem 1 is as follows. We consider the composition of k blow-ups $f: X \to \mathbf{P}^2$ satisfying that k is minimal in order that the divisor D has normal crossings, where D is the reduced divisor obtained from $f^*(C)$. Then we note that $K + D$ is numerically equivalent to an effective rational divisor if and only if the logarithmic Kodaira dimension $\overline{\kappa}(\mathbf{P}^2 - C) \geq 0$ [**3**, §11.2]. Making use of the above result when $\overline{\kappa}(\mathbf{P}^2 - C) \geq 0$, we shall prove Proposition 4 below, from which Theorem 1 will be easily deduced.

3. Proofs. We denote by $2(m), 3(n)$ and $3(n) + 2$ the sequences $(2, \ldots, 2)$, $(3, \ldots, 3)$ and $(3, \ldots, 3, 2)$ respectively, where 2 appears m times in the first sequence and 3 appears n times in the latter two sequences. For a cusp P on C the sequence of the multiplicities of all the infinitely near singular points of P will be called the sequence of P for short. Let P_i be the cusp on C with multiplicity 2, where $i = 1, \ldots, r$, and let $2(m_i)$ be the sequence of it. Then the singularity at P_i is analytically equivalent to the one at (0,0) defined by $y^2 + x^{2m_i+1} = 0$. Hence the Milnor number of the singularity at P_i is $2m_i$. On the other hand, let Q be the cusp with multiplicity 3. Then there are two cases, i.e., the sequence of Q is $3(n)$ or $3(n) + 2$ for some $n \geq 1$. Let Q_i be the cusp with the sequence $3(n_i)$ [resp. $3(n_i) + 2$], where $i = 1, \ldots, s$ [resp. $i = s + 1, \ldots, s + t$].

LEMMA 2. *The singularity at Q_i is equivalent to the one at* $(0,0)$ *defined by* $y^3 + a(x)y + x^{N_i} = 0$, *where* $N_i = 3n_i + 1$ *for* $i = 1, \ldots, s$ [*resp.* $N_i = 3n_i + 2$ *for* $i = s + 1, \ldots, s + t$] *and* $a(x)$ *is convergent power series with the order* $\geq 2n_i + 1$ [*resp.* $2n_i + 2$].

PROOF. Applying the Weierstrass preparation theorem and next doing a Tschirnhaus transformation, we can put the local equation of C at Q_i into the form $y^3 + a(x)y + b(x) = 0$, where $a(x)$ and $b(x)$ are convergent power series. Since the singularity is cuspidal, by doing blow-ups at the infinitely near singular points of Q_i, we infer that $\operatorname{ord} a(x) \geq 2n_i + 1$ [resp. $2n_i + 2$] and $\operatorname{ord} b(x) = 3n_i + 1$ [resp. $3n_i + 2$]. Then, by taking new coordinates, we arrive at the conclusion.

We put $m = \sum_{i=1}^{r} m_i$ and $n = \sum_{i=1}^{s+t} n_i$. From the above lemma the following one is obtained by a simple calculation.

LEMMA 3. *The Milnor number of the singularity at Q_i is $6n_i$, where $i = 1, \ldots, s$* [*resp.* $6n_i + 2$, *where* $i = s + 1, \ldots, s + t$]. *Hence* $\sigma = \sum_{i=1}^{r} 6[m_i/3] + 6n$.

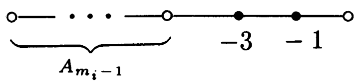
The "number" of the singular points of C is $m + n + t$. We have the following estimate.

PROPOSITION 4. $2d^2 - 3d > 5m + 14n + s + 6t$.

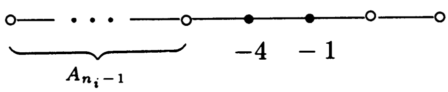
From this proposition we infer readily the theorem, so we shall prove this one hereafter.

Let $f: X \to \mathbf{P}^2$ be the composition of k blow-ups such that k is minimal in order that the divisor D has normal crossings, where D is the reduced divisor obtained from $f^*(C)$. Then the dual graphs of (i) $f^{-1}(P_i)$, (ii) $f^{-1}(Q_i)$ for $i = 1, \ldots, s$, and (iii) $f^{-1}(Q_i)$ for $i = s + 1, \ldots, s + t$, are described as follows respectively, where $\circ$ denotes the curve with the self-intersection number -2, and the number beside a curve indicates the self-intersection number.

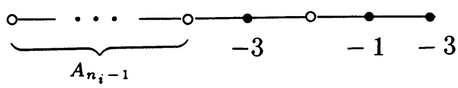
(i)



(ii)



(iii)



Thus $k = m + n + 2r + 3s + 3t$. The topological Euler characteristic $e(X) = 3 + k$ and $e(D) = 2 - 2g + k$. Let C' be the proper transform of C by f^{-1}, then the self-intersection number C'^2 is $d^2 - (4m + 2r + 9n + 3s + 6t)$. Let K be the canonical divisor on X, then $C'^2 + KC' = 2g - 2$ and $K^2 = 9 - k$. Since $D(K + D) = 2g - 2$ and $KD = s + t = KC'$, we have that $(K + D)^2 = 7 + s + t + KC' + 2g - k$. Hence the following relations hold true.

LEMMA 5. *$e(X) = 3+k$, $e(D) = 2-2g+k$ and $(K+D)^2 = 4g+5+s+t-k-C'^2$.*

We shall prove the proposition by examining the following cases separately:
(1) $r + s + t \geq 2$ or $g \geq 1$,
(2) $r+s+t = 1$ and $g = 0$. First we treat case (1). Thanks to [**6**], if C is the curve with the property (1), then $\overline{\kappa}(\mathbf{P}^2 - C) \geq 0$. Hence, applying Lemma 1 and noting that D is not a semistable curve, we infer from the above results that $d^2 + 2g > 3m + 8n + s + 4t + 2$. Using the genus formula $2g = (d - 1)(d - 2) - 2m - 6n - 2t$, we arrive at the inequality of Proposition 4.

Next we treat the case (2). Let P be the unique cusp and put $e = \mathrm{mult}_P C$. In case $d \geq 3e$, then $\overline{\kappa}(\mathbf{P}^2 - C) = 2$ [**7**, Proposition 1]. So that the proof is the same as in the case (1). On the contrary, in case $d < 3e$, then the validity of the inequality is checked directly by using the genus formula. Thus the proof of Proposition 4 is complete.

Putting the proposition and the genus formula together, we get the following inequality.

PROPOSITION 6. *$14g > d^2 - 12d + 14 + m + 3s + 4t$.*

Then Corollary 2 is clear.

4. Relevant results. In case d is a multiple of 3, i.e., $d = 3h$, we consider the minimal resolution S of the triple covering of $\mathbf{P}^2$ branched along C. Observing the Picard number of S, we can show the following.

REMARK 7. $5\sigma \leq 39h^2 - 27h + 6$ and $10g \geq 6h^2 - 18h + 4$.

Note that, if $h \leq 17$, then the former inequality is better than the one in Theorem 1.

Now here is a conjecture.

CONJECTURE 8. If $C - \{P\} \cong \mathbf{A}^1$, then $d < 3e$, where $e = \mathrm{mult}_P C$.

In case $e = 2$, this conjecture holds true [**8**]. If $e = 3$, then $d \leq 10$ by Proposition 6. Moreover by Remark 7 we see that $d \neq 9$. So it remains to be proved that $d \neq 10$.

Finally we present an example, which is proved by simple but laborious computations.

EXAMPLE 9. Let C be as in the above conjecture. Suppose that $d = 6$ and $e = 3$. Then the sequence of P is $3(3) + 2$ and $\sigma = 18$. The curve C is projectively equivalent to C_t for some $t \in \mathbf{C}$, which is defined by

$$(y - x^2)^3 + t(y - x^2)y^4 + xy^5 = 0.$$

Two curves C_t and C_s are projectively equivalent if and only if $t^5 = s^5$.

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