PLANE CURVES WHOSE SINGULAR POINTS ARECUSPS

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ABSTRACT. Let $C$ be an irreducible curve of degree $d$ in the complex projective plane. We assume that each singular point is a one place point with multiplicity 2 or 3. Let $\sigma$ be the sum of "the Milnor numbers" of the singularities. Then we shall show that $7\sigma < 6d^2 - 9d$. This gives a necessary condition for the existence of such a curve, for example, if $C$ is rational, then $d \leq 10$.

1. Introduction. Let $C$ be an irreducible curve of degree $d$ in the complex projective plane $\mathbb{P}^2$. We assume that $C$ is not smooth and each singular point is a cusp (i.e., one place point) with multiplicity 2 or 3. Let $P_i$ be the singular point and $\mu_i$ be the Milnor number of the singularity at $P_i$, where $i = 1, 2, \ldots$. Then, putting $\sigma = \sum_i 6[\mu_i/6]$, where $[\ ]$ denotes the Gauss' symbol, we have the following inequality (cf. [1, §5]).

**Theorem 1.** Let $C$ be the above-mentioned curve. Then $7\sigma < 6d^2 - 9d$.

There has been a problem whether there exist curves in $\mathbb{P}^2$ with assigned numerical characters satisfying the genus formula of Clebsch [3, §9.1]. More than half a century ago Lefschetz and Zariski studied such a problem for Plückerian characters [4, 9]. Now, let $g$ be the genus of the normalization of $C$. Then from the above theorem we obtain the following inequality.

**Corollary 2.** $14g \geq d^2 - 12d + 16$. Especially, if $C$ is rational, then $3 \leq d \leq 10$.

**Remark 3.** The genus formula implies $\limsup_{d \to \infty} \sigma/d^2 \leq 1$, but the above theorem gives a better inequality $\limsup_{d \to \infty} \sigma/d^2 \leq 6/7$ (cf. [2, §8]).

2. Miyaoka's inequality. The key to the proof of the theorem is a result of Miyaoka [5, Corollary 1.2]. Before stating it we list the necessary notations.

- $X$: a projective nonsingular surface,
- $D$: a reduced divisor on $X$ with normal crossings,
- $K$: the canonical divisor on $X$,
- $e(V)$: the topological Euler characteristic of a variety $V$. Then the result is stated as follows.

**Lemma 1.** If $K + D$ is numerically equivalent to an effective rational divisor, then $3\{e(X) - e(D)\} \geq (K + D)^2$. If the equality holds, then $D$ is a semistable curve.
The outline of the proof of Theorem 1 is as follows. We consider the composition of $k$ blow-ups $f: X \to \mathbb{P}^2$ satisfying that $k$ is minimal in order that the divisor $D$ has normal crossings, where $D$ is the reduced divisor obtained from $f^*(C)$. Then we note that $K + D$ is numerically equivalent to an effective rational divisor if and only if the logarithmic Kodaira dimension $\kappa(\mathbb{P}^2 - C) \geq 0$ [3, §11.2]. Making use of the above result when $\kappa(\mathbb{P}^2 - C) \geq 0$, we shall prove Proposition 4 below, from which Theorem 1 will be easily deduced.

3. Proofs. We denote by $2(m), 3(n)$ and $3(n) + 2$ the sequences $(2, \ldots, 2)$, $(3, \ldots, 3)$ and $(3, \ldots, 3, 2)$ respectively, where $2$ appears $m$ times in the first sequence and $3$ appears $n$ times in the latter two sequences. For a cusp $P$ on $C$ the sequence of the multiplicities of all the infinitely near singular points of $P$ will be called the sequence of $P$ for short. Let $P_i$ be the cusp on $C$ with multiplicity 2, where $i = 1, \ldots, r$, and let $2(m_i)$ be the sequence of it. Then the singularity at $P_i$ is analytically equivalent to the one at $(0,0)$ defined by $y^2 + x^{2m_i} = 0$. Hence the Milnor number of the singularity at $P_i$ is $2m_i$. On the other hand, let $Q$ be the cusp with multiplicity 3. Then there are two cases, i.e., the sequence of $Q$ is $3(n)$ or $3(n) + 2$ for some $n \geq 1$. Let $Q_i$ be the cusp with the sequence $3(n_i)$ [resp. $3(n_i) + 2$], where $i = 1, \ldots, s$ [resp. $i = s + 1, \ldots, s + t$].

**LEMMA 2.** The singularity at $Q_i$ is equivalent to the one at $(0,0)$ defined by $y^3 + a(x)y + x^{N_i} = 0$, where $N_i = 3n_i + 1$ for $i = 1, \ldots, s$ [resp. $N_i = 3n_i + 2$ for $i = s + 1, \ldots, s + t$] and $a(x)$ is convergent power series with the order $\geq 2n_i + 1$ [resp. $2n_i + 2$].

**PROOF.** Applying the Weierstrass preparation theorem and next doing a Tschirnhaus transformation, we can put the local equation of $C$ at $Q_i$ into the form $y^3 + a(x)y + b(x) = 0$, where $a(x)$ and $b(x)$ are convergent power series. Since the singularity is cuspidal, by doing blow-ups at the infinitely near singular points of $Q_i$, we infer that $\text{ord} a(x) \geq 2n_i + 1$ [resp. $2n_i + 2$] and $\text{ord} b(x) = 3n_i + 1$ [resp. $3n_i + 2$]. Then, by taking new coordinates, we arrive at the conclusion.

We put $m = \sum_{i=1}^{r} m_i$ and $n = \sum_{i=1}^{s+t} n_i$. From the above lemma the following one is obtained by a simple calculation.

**LEMMA 3.** The Milnor number of the singularity at $Q_i$ is $6n_i$, where $i = 1, \ldots, s$ [resp. $6n_i + 2$, where $i = s + 1, \ldots, s + t$]. Hence $\sigma = \sum_{i=1}^{s+t} 6\lfloor m_i/3 \rfloor + 6n$.

The "number" of the singular points of $C$ is $m + n + t$. We have the following estimate.

**PROPOSITION 4.** $2d^2 - 3d > 5m + 14n + s + 6t$.

From this proposition we infer readily the theorem, so we shall prove this one hereafter.

Let $f: X \to \mathbb{P}^2$ be the composition of $k$ blow-ups such that $k$ is minimal in order that the divisor $D$ has normal crossings, where $D$ is the reduced divisor obtained from $f^*(C)$. Then the dual graphs of (i) $f^{-1}(P_i)$, (ii) $f^{-1}(Q_i)$ for $i = 1, \ldots, s$, and (iii) $f^{-1}(Q_i)$ for $i = s + 1, \ldots, s + t$, are described as follows respectively, where $\circ$ denotes the curve with the self-intersection number $-2$, and the number beside a curve indicates the self-intersection number.
Thus $k = m + n + 2r + 3s + 3t$. The topological Euler characteristic $e(X) = 3 + k$ and $e(D) = 2 - 2g + k$. Let $C'$ be the proper transform of $C$ by $f^{-1}$, then the self-intersection number $C'^2$ is $d^2 - (4m + 2r + 9n + 3s + 6t)$. Let $K$ be the canonical divisor on $X$, then $C'^2 + KC' = 2g - 2$ and $K^2 = 9 - k$. Since $D(K + D) = 2g - 2$ and $KD = s + t = KC'$, we have that $(K + D)^2 = 7 + s + t + KC' + 2g - k$. Hence the following relations hold true.

**Lemma 5.** $e(X) = 3 + k$, $e(D) = 2 - 2g + k$ and $(K + D)^2 = 4g + 5 + s + t - k - C'^2$.

We shall prove the proposition by examining the following cases separately:

1. $r + s + t \geq 2$ or $g \geq 1$,
2. $r + s + t = 1$ and $g = 0$. First we treat case (1). Thanks to [6], if $C$ is the curve with the property (1), then $\kappa(P^2 - C) \geq 0$. Hence, applying Lemma 1 and noting that $D$ is not a semistable curve, we infer from the above results that $d^2 + 2g > 3m + 8n + s + 4t + 2$. Using the genus formula $2g = (d - 1)(d - 2) - 2m - 6n - 2t$, we arrive at the inequality of Proposition 4.

Next we treat the case (2). Let $P$ be the unique cusp and put $e = \text{mult}_PC$. In case $d \geq 3e$, then $\kappa(P^2 - C) = 2$ [7, Proposition 1]. So that the proof is the same as in the case (1). On the contrary, in case $d < 3e$, then the validity of the inequality is checked directly by using the genus formula. Thus the proof of Proposition 4 is complete.

Putting the proposition and the genus formula together, we get the following inequality.

**Proposition 6.** $14g > d^2 - 12d + 14 + m + 3s + 4t$.

Then Corollary 2 is clear.

4. Relevant results. In case $d$ is a multiple of 3, i.e., $d = 3h$, we consider the minimal resolution $S$ of the triple covering of $P^2$ branched along $C$. Observing the Picard number of $S$, we can show the following.

**Remark 7.** $5\sigma \leq 39h^2 - 27h + 6$ and $10g \geq 6h^2 - 18h + 4$.

Note that, if $h \leq 17$, then the former inequality is better than the one in Theorem 1.

Now here is a conjecture.

**Conjecture 8.** If $C - \{P\} \cong A^1$, then $d < 3e$, where $e = \text{mult}_PC$.

In case $e = 2$, this conjecture holds true [8]. If $e = 3$, then $d \leq 10$ by Proposition 6. Moreover by Remark 7 we see that $d \neq 9$. So it remains to be proved that $d \neq 10$. 

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Finally we present an example, which is proved by simple but laborious computations.

**EXAMPLE 9.** Let $C$ be as in the above conjecture. Suppose that $d = 6$ and $e = 3$. Then the sequence of $P$ is $3(3) + 2$ and $\sigma = 18$. The curve $C$ is projectively equivalent to $C_t$ for some $t \in C$, which is defined by

$$(y - x^2)^3 + t(y - x^2)y^4 + xy^5 = 0.$$ 

Two curves $C_t$ and $C_s$ are projectively equivalent if and only if $t^5 = s^5$.

**References**