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REDUCTION OF A MATRIX DEPENDING ON PARAMETERS TO A DIAGONAL FORM BY ADDITION OPERATIONS

L. N. VASERSTEIN

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ABSTRACT. It is shown that any n by n matrix with determinant 1 whose entries are real or complex continuous functions on a finite dimensional normal topological space can be reduced to a diagonal form by addition operations if and only if the corresponding homotopy class is trivial, provided that $n \neq 2$ for real-valued functions; moreover, if this is the case, the number of operations can be bounded by a constant depending only on n and the dimension of the space. For real functions and $n = 2$, we describe all spaces such that every invertible matrix with trivial homotopy class can be reduced to a diagonal form by addition operations as well as all spaces such that the number of operations is bounded.

Introduction. Let X be a topological space $\mathbf{R}^X$ the ring of all continuous functions $X \to \mathbf{R}$ (the reals), $\mathbf{R}_0^X$ the subring of bounded functions. For any natural number n and a ring A, $M_n A$ denotes the ring of all n by n matrices over A.

A matrix α in $M_n\mathbf{R}^X$ can be regarded as a real matrix depending continuously on a parameter which ranges over X, or as a continuous map $X \to M_n\mathbf{R}$.

Assume now that $\det(\alpha) = 1$, i.e. $\alpha \in \mathrm{SL}_n\mathbf{R}^X$. We want to reduce α to the identity matrix 1_n by addition operations, i.e. represent α as a product of elementary matrices a^{ij}, where $a \in A = \mathbf{R}^X$, $1 \leq i \neq j \leq n$. Since the subgroup E_nA of SL_nA generated by all elementary matrices is normal [**6**], it does not matter whether we use row or column addition operations, or both. Note that, by the Whitehead lemma, every diagonal matrix in SL_nA is a product of $4(n-1)$ elementary matrices (for any commutative ring A), so a matrix α in SL_nA, can be reduced to 1_n if and only if it can be reduced to a diagonal form.

When X is a point, so $A = \mathbf{R}^X = \mathbf{R}$, it is well known that this can be done. Moreover [**3**, Remark 10 with $\mathrm{sr}(\mathbf{R}) = m = 1$], this can be done using at most $(n-1)(3n/2+1)$ addition operations.

For an arbitrary X, a homotopy obstruction may exist which prevents the reduction. Namely, the addition operations do not change the homotopy class $\pi(\alpha)$ of the corresponding map $X \to \mathrm{SL}_n\mathbf{R}$. So if this class is not trivial, the reduction is impossible.

Assume now that the homotopy class $\pi(\alpha)$ is trivial (for example, this is always the case when X is contractible). Is it possible to reduce α to 1_n by addition operations, i.e. does α belong to the subgroup $E_n\mathbf{R}^X$ of $\mathrm{SL}_n\mathbf{R}^X$ generated by elementary matrices)? If yes, how many operations are needed?

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In this paper, we give an answer to both questions. It turns out that the answer in case $n = 2$ is different from that in the case $n \neq 2$. The reason is that the fundamental group $\pi_1(\mathrm{SL}_n\mathbf{R})$ is infinite when $n = 2$ (namely, it is infinite cyclic) and it is finite otherwise (it is of order 2 when $n \geq 3$).

More precisely, for any α in $\mathrm{SL}_n A$ (where A is a commutative ring with 1 such as $A = \mathbf{R}^X$ or $\mathbf{R}_0^X$), denote by $l_A(\alpha)$ the least k such that α is a product of k elementary matrices over A. If no such k exists, i.e. α is outside E_nA, we set $l_A(\alpha) = \infty$. As in [**3**], $e_n(A)$ denotes the supremum of $l_A(\alpha)$, where α ranges over E_nA.

THEOREM 1. *Let X be a topological space, $A = \mathbf{R}^X$ or $\mathbf{R}_0^X$ as above. Then* (a) *$e_2(A) < \infty$ if and only if $\mathbf{R}^Y = \mathbf{R}$ for every connected component Y of X*; (b) *$l_A(\alpha) < \infty$ for all α in SL_2A with $\pi(\alpha) = 0$ if and only if X is pseudocompact, i.e. $\mathbf{R}^X = \mathbf{R}_0^X$.*

Now we consider the case $n \geq 3$.

THEOREM 2. *For any integers $n \geq 3$ and $d \geq 0$ there is a natural number z such that $l_A(\alpha) \leq z$ for $A = \mathbf{R}^X$ or $\mathbf{R}_0^X$ with any normal topological space X of dimension d and any α in SL_nA with $\pi(\alpha) = 0$. In particular, $e_n(A) \leq z$.*

As a consequence of Theorem 2 (which is extracted here from results of [**1**, **2**]) we obtain that SL_nA/E_nA is a homotopy type invariant of X for finite dimensional spaces X if $n \geq 3$. This was proved in [**6**] for $X = \mathbf{R}$ and in [**4**] for $X = \mathbf{R}^3$ by different methods.

It is easy to extend Theorems 1 and 2 to subrings A of $\mathbf{R}^X$ different from $\mathbf{R}^X$ and $\mathbf{R}_0^X$, compare with [**6**]. This is because of the following fact.

PROPOSITION 3. *Let A be as in Theorem 1 and B is a subring with 1 of A such that B is dense in A and GL_1B is open in B, both in the topology of uniform convergence. Then $|e_n(B) - e_n(A)| \leq (n+3)(n-1)$ for every n.*

Note that the condition that GL_1B is open in B, i.e. $fB = B$ for every function f in B sufficiently close to 1, cannot be dropped. The following example shows this. Let X be the unit interval $0 \leq x \leq 1$ and $B = \mathbf{R}[x]$, the polynomial ring. In this example, $\mathrm{SL}_nB = E_nB$ for all n, but $e_n(B) = \infty$ for each $n \geq 2$ [**5**]. At the same time, B is dense in $A = \mathbf{R}^X = \mathbf{R}_0^X$ and $e_n(A) < \infty$ for $n \geq 3$ by Theorem 1.

Next we consider the ring $\mathbf{C}^X$ of all continuous functions $X \to \mathbf{C}$, the complex numbers, and its subring $\mathbf{C}_0^X$ of bounded functions.

THEOREM 4. *For any natural number n and an integer $d \geq 0$ there is a natural number z' such that $l_A(\alpha) \leq z'$ for any normal topological space X of dimension d and any matrix α in SL_nA with $\pi(\alpha) = 0$, where $A = \mathbf{C}^X$ or $\mathbf{C}_0^X$. In particular, $e_n(A) \leq z' < \infty$ for all n.*

COROLLARY 5. *For each natural number n and an integer $d \geq 0$ there is a natural number z'' such that $e_n(B) \leq z'' < \infty$ for any dense subring B with 1 of A with GL_1B open in B, where A is as in Theorem 4.*

Note that $\mathbf{C}^X$ is endowed with the topology of the uniform convergence, and that the constant z'' depends only on n and the dimension of X. We do not give any explicit bounds in this paper, although the proofs in [**1**, **2**] seem to be constructive enough to yield some explicit bounds.

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PROOF OF THEOREM 1. Let X be a topological space, $A = \mathbf{R}^X$ or $\mathbf{R}_0^X$ as in Theorem 1. For any $f \in \mathbf{R}^X$ we set

$$\rho f = \begin{pmatrix} \cos(f) & \sin(f) \\ -\sin(f) & \cos(f) \end{pmatrix} \in SO_2\mathbf{R}^X = SO_2\mathbf{R}_0^X \subset SL_2\mathbf{R}_0^X.$$

For any $f \in \mathbf{C}^X$, let $||f|| = \sup|f(x)|$, where x ranges over X.

LEMMA 6. *Let α be a product of k elementary matrices in SL_2A with $k \geq 1$. Then α has the form $\delta\varepsilon(\rho f)$, where ε is an elementary matrix, δ is a diagonal matrix, $f \in \mathbf{R}^X$ and $||f|| \leq (k-1)\pi/2$.*

PROOF. We proceed by induction on k. When $k = 1$, we can take $\delta = 1_2$, $f = 0$. Assume now that $k \geq 2$ and $\alpha = \varepsilon_1 \cdots \varepsilon_k$ with elementary matrices ε_i. By the induction hypothesis, $\varepsilon_2 \cdots \varepsilon_k = \delta'\varepsilon'(\rho(f'))$ with an elementary ε', diagonal δ', and $||f'|| \leq (k-2)\pi/2$. If the elementary matrices ε_1 and ε' are of the same type, i.e. $\varepsilon_1\varepsilon'$ is an elementary matrix, then $\alpha = \delta'(\delta'^{-1}\varepsilon_1\delta'\varepsilon')(\rho(f'))$ is the required representation, i.e. we can take $\delta = \delta'$, $\varepsilon = \delta^{-1}\varepsilon_1\delta\varepsilon'$, $f = f'$. Assume now that $\varepsilon', \varepsilon_1$ are not of the same type, that is either $\varepsilon_1 \in A^{1,2}$ and $\varepsilon' \in A^{2,1}$, or $\varepsilon_1 \in A^{2,1}$ and $\varepsilon' \in A^{1,2}$. Consider the first case (the second one is similar).

Then $\delta'^{-1}\varepsilon_1\delta'\varepsilon' = b^{1,2}c^{2,1}$ with b and c in A. Applying the Gram-Schmidt process to the rows of this matrix, we obtain

$$b^{1,2}c^{2,1} = \begin{pmatrix} 1+bc & b \\ c & 1 \end{pmatrix} = \begin{pmatrix} 1/e & 0 \\ 0 & e \end{pmatrix} \begin{pmatrix} e(1+bc) & eb \\ c/e & 1/e \end{pmatrix}$$

with $e = (1+c^2)^{1/2} \geq 0$,

$$\begin{pmatrix} e(1+bc) & eb \\ c/e & 1/e \end{pmatrix} = \begin{pmatrix} 1 & b+c+cbc \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/e & -c/e \\ c/e & 1/e \end{pmatrix} = (b+c+cbc)^{1,2}\rho f''$$

with $(c/e, 1/e) = (-\sin(f''), \cos(f''))$ and $||f''|| < \pi/2$.

Thus, $\alpha = \varepsilon_1 \cdots \varepsilon_k = \varepsilon_1\delta'\varepsilon'(\rho(f')) = \delta'(\delta'^{-1}\varepsilon_1\delta')\varepsilon'(\rho f') = \delta'(b^{1,2}c^{2,1})\rho f' = \delta\varepsilon\rho f$, where

$$\delta = \delta' \begin{pmatrix} 1/e & 0 \\ 0 & e \end{pmatrix}$$

is a diagonal matrix, $\varepsilon = (b+c+cbc)^{1,2}$ is an elementary matrix, and $f = f' + f''$ with $||f|| \leq ||f'|| + ||f''|| \leq (k-2)\pi/2 + \pi/2 = (k-1)\pi/2$.

Lemma 6 is proved.

COROLLARY 7. *If X is connected, then for any $g \neq 0$ in A we have*

$$l_A(\rho g) \geq (\sup(g) - \inf(g))/\pi + 1.$$

PROOF. Suppose that ρg is a product of k elementary matrices. Then by Lemma 6, $\rho g = \delta\varepsilon\rho f$ with diagonal δ, elementary ε, and $||f|| \leq (k-1)\pi/2$. It follows that $\varepsilon = 1_2$, and $\delta = 1_2$ or -1_2, hence $f - g = 2\pi m$ or $\pi + 2\pi m$ with a continuous function m: $X \to \mathbf{Z}$ (the integers). Since X is connected, m is a constant. Therefore, $\sup(g) - \inf(g) = \sup(f) - \inf(f) \leq 2||f|| \leq (k-1)\pi$. Thus, $k \geq 1 + (\sup(f) - \inf(f))/\pi = (\sup(g) - \inf(g))/\pi + 1$. The corollary is proved.

PROPOSITION 8. *For any f in A, we have $l_A(\rho f) \leq 2(\sup(f) - \inf(f))/\pi + 6$.*

PROOF. If f is not bounded, there is nothing to prove. So we can assume that f is bounded, i.e. $f \in \mathbf{R}_0^X$, i.e. $r = \sup(f) - \inf(f) < \infty$. Set $t = (\sup(f) + \inf(f))/2$ and write $f = t + (f - t)$, where t means a constant function and $|f - t| \leq r/2$ everywhere on X. We have to write ρf as a product of $k \leq 2r/\pi + 6$ elementary matrices over A. Set $s = [r/\pi + 1]$.

We have $\rho f = \rho t(\rho((f-t)/s))^s$. Note that $|(f-t)/s| \leq r/2s = r/2([r/\pi + 1]) < \pi/2$. So $\cos((f-t)/s) \in \mathrm{GL}_1 A$. Therefore $\rho((f-t)/s)$ is a product of two elementary matrices and a diagonal matrix, hence ρf is a product of $\rho(t)$, a diagonal matrix, and $2s$ elementary matrices. The product of the constant matrix ρt and the diagonal matrix has an invertible entry in the first column, so it is the product of at most 4 elementary matrices. Thus, ρf is the product of $2s + 4 \leq 2r/\pi + 2 + 4 = 2r/\pi + 6$ elementary matrices. This proves the proposition.

Now we are prepared to prove Theorem 1. The case of empty X is trivial, so let X be nonempty.

To prove part (a) of Theorem 1, suppose first that $\mathbf{R}^Y = \mathbf{R}$ for every connected component Y of X. Then $\mathbf{R}^X = \mathbf{R}^{X'}$ and $\mathbf{R}_0^X = \mathbf{R}_0^{X'}$, where X' is the discrete set of connected components of X. So $e_2(A) = e_2(\mathbf{R}) = 4 < \infty$.

Suppose now that $\mathbf{R}^Y \neq \mathbf{R}$ (or, equivalently $\mathbf{R}_0^Y \neq \mathbf{R}$), for some connected component Y of X. We will show that then $e_2(B) = \infty$ for $B = \mathbf{R}^Y$ and for $B = \mathbf{R}_0^Y$. This will imply that $e_2(A) = \infty$.

Pick a nonconstant function f in $\mathbf{R}^Y$. By Corollary 7 applied to Y instead of X, $l_B(\rho(fm)) \geq m(\sup(f) - \inf(f))/\pi + 1$ for any natural number m. Taking large m, we conclude that $e_2(B) = \infty$.

To prove Theorem 1(b), consider the exact sequence [**6**] (see also [**1**, **2**])

$$0 \to \mathbf{R}^X/\mathbf{R}_0^X \to \mathrm{SL}_2 A/E_2 A \to \pi^1(X) \to 0.$$

The sequence says that X is pseudocompact, if and only if $\mathrm{SL}_2 A/E_2 A = \pi^1(X)$, i.e. if and only if $\alpha \in E_2 A$ for every α in $\mathrm{SL}_2 A$ with $\pi(\alpha) = 0$.

PROOF OF THEOREM 2. If the theorem is wrong, then for some $n \geq 3$ and $d \geq 0$ there is a sequence $X(i)$ of normal topological spaces of dimension d and $\alpha(i) \in \mathrm{SL}_n A(i)$, where $A(i) = \mathbf{R}^{X(i)}$ or $\mathbf{R}_0^{X(i)}$ depending on whether $A = \mathbf{R}^X$ or $\mathbf{R}_0^X$ in the theorem, such that $\pi(\alpha(i)) = 0$ for all i and $l_{A(i)}(\alpha(i)) \to \infty$. In the case $A = \mathbf{R}_0^X$, we can bring each $\alpha(i)$ to $\mathrm{SO}_n A(i)$ by $(n+6)(n-1)/2$ addition operations [**6**, Lemma 21], so we can assume that $\alpha(i) \in \mathrm{SO}_n A(i)$.

We define X to be the disjoint union of all $X(i)$. The matrices $\alpha(i) \in \mathrm{SL}_n A(i)$ give a matrix α in $\mathrm{SL}_n A$ whose restriction to $X(i)$ is $\alpha(i)$. We have $\pi(\alpha) = 0$, i.e. the corresponding map $X \to \mathrm{SL}_n \mathbf{R}$ is homotopic to the trivial map $X \to 1_n$. Now we invoke results of [**1**, **2**] to conclude that the map $X \to \mathrm{SL}_n \mathbf{R}$ is *uniformly* homotopic to the trivial map.

First of all, Gram-Schmidt's process [**6**] reduces the matrix α in $\mathrm{SL}_n A$ (as well as its homotopy to the trivial map) to a matrix f: $X \to \mathrm{SO}_n \mathbf{R}$ in the special orthogonal group (of the sum of n squares) $\mathrm{SO}_n A$ by addition operations (resp. to a homotopy of f to the trivial map into this subgroup). Since $n \geq 3$, the fundamental group $\pi_1 \mathrm{SO}_n \mathbf{R} = \mathbf{Z}/2\mathbf{Z}$ is finite. Since X is finite dimensional and normal, Theorem 1 of [**1**] (see [**2**] for a shorter and a great deal more transparent proof) gives the desired conclusion.

Thus, f is *uniformly* homotopic to the trivial map in $\mathrm{SO}_n \mathbf{R}$, i.e. the corresponding matrix in $\mathrm{SO}_n A$ belongs to the connected component of 1_n, hence α belongs to the connected component of 1_n in $\mathrm{SL}_n A$, where $\mathrm{SL}_n A$ is endowed with the topology induced by the uniform convergence topology on A.

It is known (see, for example, [**6**, Theorem 2]) that this component coincides with $E_n A$. So α is a product of (finitely many) elementary matrices. Restriction to $X(i)$ yields that each $\alpha(i)$ is the product of a bounded (uniformly in i) number of elementary matrices over $A(i)$. This contradicts to our choice of $\alpha(i)$ with $l_{A(i)}(\alpha(i)) \to \infty$. So Theorem 2 is proved.

REMARK. The condition that X is normal can be easily dropped; for arbitrary X, the dimension should be understood in the sense of [**7**], i.e. it is $sr(A) - 1$. It is not clear how z depends on d or whether a uniform upper bound exists. Obviously, z cannot be taken less than $n^2 - 1$, the dimension of $\mathrm{SL}_n \mathbf{R}$.

PROOF OF PROPOSITION 3.

LEMMA 9. *Let B be a commutative topological ring with* 1 *such that* $\mathrm{GL}_1 B$ *is open in B. Then $l_B(\alpha) \leq (n+3)(n-1)$ for any n and any matrix α in* $\mathrm{SL}_n B$ *sufficiently close to* 1_n.

PROOF. It is clear that every α sufficiently close to 1_n has the form $\beta\gamma$ with a lower triangular β with ones along the main diagonal and an upper triangular matrix γ. We have $l_B(\beta) \leq n(n-1)/2$, and $l_B(\gamma) \leq (n+6)(n-1)/2$ by [**6**, Lemma 21]. So $l_B(\alpha) \leq n(n-1)/2 + (n+6)(n-1)/2 = (n+3)(n-1)$.

Let us prove now Proposition 3. Let $\alpha \in E_n B$. We can write α as a product of $k = l_A(\alpha)$ elementary matrices over A. Using that B is dense in A we can write α as a product of k elementary matrices over B and a matrix α' arbitrarily close to 1_n. By Lemma 9, α' is a product of $(n+3)(n-1)$ elementary matrices. So $l_A(\alpha) \leq l_B(\alpha) \leq l_A(\alpha) + (n+3)(n-1)$ for any α in $E_n B$. Therefore, $e_n(B) \leq e_n(A) + (n+3)(n-1)$.

Let now $\alpha \in E_n A$. Since B is dense in A, we can write $\alpha = \beta\gamma$ with $\beta \in E_n B$ and γ arbitrarily close to 1_n. So $l_A(\alpha) \leq l_A(\beta) + (n+3)(n-1)$, by Lemma 9 applied to A instead of B. So, $e_n(A) \leq e_n(B) + (n+3)(n-1)$.

Proposition 3 is proved.

PROOF OF THEOREM 4. If the theorem is wrong, then for some $n \geq 2$ there is a sequence $X(i)$ of normal topological spaces of dimension d and $\alpha(i) \in \mathrm{SL}_n A(i)$, where $A(i) = \mathbf{C}^{X(i)}$ or $\mathbf{C}_0^{X(i)}$ depending on whether $A = \mathbf{C}^X$ or $\mathbf{C}_0^X$ in the theorem, such that $\pi(\alpha(i)) = 0$ for all i and $l_{A(i)}(\alpha(i)) \to \infty$. In the case $A = \mathbf{C}_0^X$, we can bring each $\alpha(i)$ to $\mathrm{SU}_n A(i)$ by $(n+6)(n-1)/2$ addition operations [**6**, Lemma 21], so we can assume that $\alpha(i) \in \mathrm{SU}_n A(i)$.

We define X to be the disjoint union of all $X(i)$. The matrices $\alpha(i) \in \mathrm{SL}_n A(i)$ give a matrix α in $\mathrm{SL}_n A$ whose restriction to $X(i)$ is $\alpha(i)$. We have $\pi(\alpha) = 0$. By [**1**, **2**], α belongs to the connected component of 1_n, where $\mathrm{SL}_n A$ is endowed with the topology induced by the uniform convergence topology on A (here we used that $\pi_1(\mathrm{SL}_n\mathbf{R})$ is trivial).

It is known that this component coincides with E_nA. So α is a product of (finitely many) elementary matrices. Restriction to $X(i)$ yields that each $\alpha(i)$ is the product of a bounded (uniformly in i) number of elementary matrices over $A(i)$. This contradicts our choice of $\alpha(i)$ with $l_{A(i)}(\alpha(i)) \to \infty$.

So Theorem 4 is proved.

PROOF OF COROLLARY 5. In the case $A = \mathbf{C}^X$, we argue as in the proof of Proposition 3 to conclude that $l_A(\alpha) \le l_B(\alpha) \le l_A(\alpha) + (n+3)(n-1)$ for each α in SL_nB, hence $e_n(B) \le e_n(A)+(n+3)(n-1)$, and that $e_n(A) \le e_n(B)+(n+3)(n-1)$. Thus, $|e_n(B) - e_n(A)| \le (n+3)(n-1)$, so we can take $z'' = z' + (n+3)(n-1)$.

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DEPARTMENT OF MATHEMATICS, PENN STATE UNIVERSITY, UNIVERSITY PARK, PENNSYLVANIA, 16802