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VON NEUMANN REGULARITY OF V-RINGS WITH ARTINIAN PRIMITIVE FACTOR RINGS

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ABSTRACT. Let R be a ring whose right primitive factor rings are artinian. It is a known result that if R is von Neumann regular, then R is a right V-ring, that is, all simple right R-modules are injective. In the present work we prove that the converse holds.

All rings we consider are associative with identity and all modules are unital. A ring R is a *right V-ring* if every simple right R-module is injective. A well-known result of Kaplansky states that a commutative ring R is (von Neumann) regular if and only if R is a V-ring. This is no longer true if one drops the commutativity condition: if V_D is an infinite-dimensional vector space over a division ring D, then $R = \mathrm{End}(V_D)$ is regular but ${}_RV$ is a simple noninjective module (see [**8**, Proposition 2.4] for a proof), hence R is not a left V-ring. On the other hand, there exist simple and noetherian V-rings which are not artinian, and hence are not regular (see [**3**]). A problem which naturally arises is to find classes of rings, containing properly the class of commutative rings, in which the two conditions of being regular and V-ring are equivalent. B. Sarath and K. Varadarajan proved in [**8**, Theorem 3.4] that the above equivalence holds for the class of rings in which all maximal right ideals are two-sided (see also [**2**, Theorem 3.7]). J. M. Goursaud and J. Valette proved in [**6**, Theorem 2.3] that if R has all primitive factor rings artinian, then R regular implies that R is a V-ring. The purpose of the present paper is to prove the converse: a right V-ring all of whose right primitive factor rings are artinian is regular.

A ring R is called *fully right idempotent* if $I = I^2$ for every right ideal I or, equivalently, if ${}_R(R/K)$ is flat for every two-sided ideal K. R is a *right Σ-V-ring* if every simple right R-module is Σ-injective. We recall that a right V-ring is fully right idempotent (see [**7**, Corollary 2.2]) and a prime fully right idempotent ring is right nonsingular (see [**2**, Lemma 4.3]). We also recall that if K is a two-sided ideal of the ring R, then ${}_R(R/K)$ is flat if and only if every injective right R/K-module is injective as an R-module (see [**1**, Proposition 5]). We shall denote with $r_R(M)$ the right annihilator of a right R-module M.

THEOREM. *Let R be a ring all of whose right primitive factor rings are artinian. Then the following conditions are equivalent:*
(1) *R is a right V-ring.*
(2) *R is a right Σ-V-ring.*
(3) *R is fully right idempotent.*
(4) *R is von Neumann regular.*

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PROOF. (1) $\Rightarrow$ (3). See [**7**, Corollary 2.2].

(3) $\Rightarrow$ (2). Assume (3), let S_R be simple and set $P = r_R(S)$.Then $_R(R/P)$ is flat and, since $S_{R/P}$ is Σ-injective, it follows that S_R is Σ-injective.

(2) $\Rightarrow$ (1) is obvious.

(4) $\Rightarrow$ (3) is clear.

(1) $\Rightarrow$ (4). Assume (1). Let us consider first the case in which R is prime and let us prove that, consequently, R is finite (Goldie) dimensional on the right. Since R is right nonsingular, then the maximal right ring of quotients Q of R is von Neumann regular. In order to prove our claim it is sufficient to show that Q is semisimple. Assume, on the contrary, that Q is not semisimple. Then there is an independent sequence $A_1, \ldots, A_n, \ldots$ of nonzero right ideals of Q. Since R is essential in Q_R, this yields an independent sequence $x_1R, \ldots, x_nR, \ldots$ of nonzero principal right ideals of R, and $x_1Q, \ldots, x_nQ, \ldots$ is still independent. Inasmuch as R is prime, there are $y_1, \ldots, y_n, \cdots \in R$ such that $x_n y_{n-1} x_{n-1} \cdot \cdots \cdot y_1 x_1 \neq 0$ for all n. For each n, let us consider the element $z_n = x_n y_{n-1} x_{n-1} \cdot \cdots \cdot y_1 x_1$ of R. We get a descending chain $Qz_1 \geq \cdots \geq Qz_n \geq \cdots$ of nonzero principal left ideals of Q. For each n there exists an idempotent $f_n \in Q$ such that $Qz_n = Qf_n$ and we have then an ascending chain $(1-f_1)Q \leq \cdots \leq (1-f_n)Q \leq \cdots$ of proper right ideals of Q. Since R is a right V-ring, there exists a maximal submodule M of Q_R such that $\bigcup_n (1-f_n)Q \subset M$ and $1 \notin M$. If $P = r_R(Q/M)$, then P is a right primitive ideal of R. Since R is fully right idempotent, then $_R(R/P)$ is flat and therefore $P = R \cap QP$; moreover $QP \subset M$.

We claim that $z_n \notin P$ for all n. For if $z_n \in P$ for some n, then $f_n \in QP \subset M$ (since $f_n = q_n z_n$ for some $q_n \in Q$) and hence $1 \in M$: a contradiction. Consequently

$$(*) \qquad x_n \notin P \quad \text{for all } n.$$

Inasmuch as Q_Q is injective, it follows from [**5**, Lemma 9.7] that there are orthogonal idempotents $e_1, \ldots, e_n, \ldots \in Q$ such that $e_nQ = x_nQ$ for all n. If some $e_n \in QP$, then $x_nR = e_n x_n R \subset QP \cap R = P$ and hence $x_n \in P$, in contradiction with $(*)$. This shows that $\bar{e}_1R, \ldots, \bar{e}_nR, \ldots$ are nonzero submodules of $(Q/QP)_R$. We claim that they are also independent. Indeed, if there are positive integers i, j with $i < j$ and $\bar{e}_j r = \bar{e}_1 r_1 + \cdots + \bar{e}_i r_i$ for some $r, r_1, \ldots, r_i \in R$, then there is $s \in QP$ such that $e_j r = e_1 r_1 + \cdots + e_i r_i + s$. Since the e_n are orthogonal, this yields $e_j r = e_j e_j r = e_j s \in QP$, that is, $\bar{e}_j r = 0$.

Now R/P is canonically a submodule of $(Q/QP)_R$ and, since each $x_n \notin P = R \cap QP$, the above argument shows that $\bar{x}_1R, \ldots, \bar{x}_nR, \ldots$ is an infinite sequence of nonzero independent submodules of $(R/P)_R$. This is impossible because R/P is artinian. We conclude that Q is semisimple and hence R_R is finite-dimensional. Finally it follows from [**7**, Lemma 3.1] that R is simple; therefore R is artinian by the hypothesis of our theorem.

If R is not prime and P is any prime ideal of R, then R/P is a prime right V-ring whose right primitive factor rings are artinian; thus R/P is artinian by the above. We see now that R satisfies the hypothesis of [**4**, Corollary 1.3] and hence R is von Neumann regular.

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