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SUBNORMAL COMPOSITION OPERATORS

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ABSTRACT. Let C be the composition operator on $L^2(X, \Sigma, m)$ given by $Cf = f \circ T$, where T is a Σ-measurable transformation from X onto X and $dm \circ T^{-1}/dm$ is strictly positive and bounded. It is shown that C is a subnormal operator if and only if the sequence $dm \circ T^{-n}/dm$ is a moment sequence for almost every point in X. Several examples of subnormal composition operators are included.

1. Introduction. Operators of the form $Cf = f \circ T$ operating on certain function spaces are called composition operators. When the underlying space is $L^2(X, \Sigma, m)$ (where m is sigma-finite), the study of such operators follows two major paths: If T is measure preserving the study of such transformations is basic to ergodic theory and the operators so defined are isometries and thus subnormal. If T is not measure preserving the operator induced by T need not be a member of any of the seminormal families of operators. References are given in §2 of this article for known characterizations of normality, quasinormality, and hyponormality of composition operators. It is then shown that certain moment sequence characterizations of subnormality of composition operators exist. These characterizations are applied to the major classes of known subnormal composition operators and an example is presented indicating the use of these characterizations in determining subnormality.

2. Subnormal composition operators. Let (X, Σ, m) be a sigma-finite measure space and suppose that T is a mapping from X onto X which is measurable (i.e. $T^{-1}\Sigma \subseteq \Sigma$) and such that $m \circ T^{-1}$ is absolutely continuous with respect to m. Let $h = h_1$ be the Radon-Nikodym derivative $dm \circ T^{-1}/dm$, and for each $n \geq 1$ let $h_n = dm \circ T^{-n}/dm$. The operator defined by $Cf = f \circ T$ is called the composition operator induced by T. A good survey article of the general properties of such operators is [**5**]. In particular C defines a bounded operator on $L^2_m = L^2(X, \Sigma, m)$ if and only if $h \in L^\infty = L^\infty(X, \Sigma)$. For the remainder of this paper we assume that $h \in L^\infty$. Recall that an operator A is *normal* if $A^*A = AA^*$, is *hyponormal* if $A^*A \geq AA^*$, and A is *subnormal* if A has an extension to a normal operator on a larger Hilbert space. Necessary and sufficient conditions for normality and hyponormality of composition operators are found in [**2**] and [**7**]. In this article we shall give several related characterizations of subnormality for composition operators. It is shown in [**2**, Theorem 9d] that if h is hyponormal then $h > 0$ a.e. Since subnormality implies hyponormality we assume that h is strictly positive throughout this article. This condition, together with the assumption that T is an onto

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mapping, implies that C is injective. This will allow us to make use of the following result.

THEOREM [**3**]. *Let A be an operator on a Hilbert space H with* kernel$(A) = \{0\}$. *Then A is subnormal if and only if for every x in H the sequence $\|A^n x\|^2$ is a moment sequence, i.e., there is an interval $I = [0, r]$ such that for each x in H one can find a Borel measure M_x such that for each $n \geq 0$,*

$$\|A^n x\|^2 = \int_I t^n \, dM_x(t).$$

An immediate application of this result to composition operators yields the following.

1. THEOREM. *C is subnormal if and only if for every $f \in L^2_m$, $\int_X h_n |f|^2 \, dm$ is a moment sequence.*

PROOF.

$$\|C^n f\|^2 = \int_X |f \circ T^n|^2 \, dm = \int_X h_n |f|^2 \, dm.$$

We shall concentrate now on determining conditions on the sequence $\{h_n\}$ in L^∞ equivalent to the subnormality condition given in Theorem 1. We shall make use of a standard characterization of moment sequences.

PROPOSITION [**10**]. *Let $\{\lambda_n\}_{n=0}^\infty$ be a sequence of positive real numbers. Then $\{\lambda_n\}$ is a moment sequence if and only if for some interval I the linear functional φ defined on the set of polynomials $P(I)$ over I by $\varphi(\sum a_n t^n) = \sum a_n \lambda_n$ is positive.*

Here "positive" means that if $p(t) \geq 0$ on I then $\varphi(p) \geq 0$. Our first characterization of subnormality for C is a functional version of the proposition above.

2. THEOREM. *Let C be a composition operator, $I = [0, \|h\|_\infty]$, and define the linear transformation L from $P(I)$ to $L^\infty(X, \Sigma)$ by*

$$L\left(\sum a_n t^n\right) = \sum a_n h_n.$$

Then C is subnormal if and only if L is positive (i.e. $p \geq 0$ on I implies $L(p) \geq 0$ a.e.).

PROOF. Suppose first that C is subnormal. Let p be a nonnegative polynomial on I and let $S = \{x \in X \colon L(p)(x) \text{ is not positive}\}$. Let S_0 be a subset of S with $m(S_0) < \infty$. By Theorem 1, since χ_{S_0} is in L^2_m, there is a measure μ on I such that for every $n \geq 0$,

$$\int_{S_0} h_n \, d_m = \int_I t^n \, d\mu.$$

It follows that $\int_{S_0} L(p) \, dm = \int_I p(t) \, d\mu \geq 0$. Thus for every measurable subset S_0 of S of finite measure $\int_{S_0} L(p) \, dm \geq 0$. It follows that $m(S) = 0$, i.e., $L(p) \geq 0$, hence L is positive.

Now suppose that L is positive. Since $L(1) = h_0 = 1$ it follows easily that L is continuous with respect to sup and essential sup norms. In particular L extends to a positive linear mapping from $C(I)$ to $L^\infty(X,\Sigma)$. Let f be a member of L^2_m. Define φ_f on $C(I)$ by

$$\varphi_f(g) = \int_X L(g)|f|^2\,dm.$$

Since $L(g)$ is in L^∞, φ_f defines a continuous linear functional on $C(I)$. Explicitly $|\varphi_f(g)| \leq (\|L\|\,\|f\|_2^2)\|g\|_\infty$. It follows from the Riesz representation theorem that there is a measure μ_f such that for all g in $C(I)$,

$$\varphi_f(g) = \int_I g\,d\mu_f,$$

i.e.,

$$\int_X L(g)|f|^2\,dm = \int_I g\,d\mu_f.$$

In particular for all $n \geq 0$,

$$\int_X h_n|f|^2\,dm = \int_I t^n\,d\mu_f.$$

It follows from Theorem 1 that C is subnormal.

It is interesting at this point to see how Theorem 2 yields the subnormality of certain well-known examples.

1. Suppose $h = 1$ a.e. (that is, T is measure preserving; equivalently C is an isometry). In this case $h_n = 1$ a.e. for each n so that $L(p) = p(1)\cdot 1$, which clearly defines a positive transformation.

2. $h = h \circ T$ (this characterizes *quasinormal* composition operators; see [**8**] and [**9**]). It is shown in [**4**] that for each n, $h_{n+1} = h \cdot E(h_n|T^{-1}\Sigma) \circ T^{-1}$, where $E(\ |T^{-1}\Sigma)$ is the conditional expectation operator. This leads immediately to the relationship $h_n = h^n$. Hence $L(\sum a_n t^n) = \sum a_n h^n$. Since $h(x) \in [0, \|h\|_\infty] = I$ a.e. L is positive.

Theorem 2 admits measure-theoretic and pointset-theoretic reformulations, as given by the following two corollaries.

3. COROLLARY. *C is subnormal if and only if for every Σ-set A of finite measure, $\{mT^{-n}(A)\}$ is a moment sequence.*

PROOF. If C is subnormal then $\int_X h_n\chi_A\,dm$ defines a moment sequence. But

$$\int_X h_n\chi_A\,dm = \int_A h_n\,dm = \int_A (dm\circ T^{-1}/dm)\,dm = mT^{-n}A.$$

Conversely, suppose for every set A of finite measure $\{mT^{-n}A\}$ is a moment sequence. Let $p(t) = \sum a_n t^n \geq 0$ on $I = [0, \|h\|_\infty]$. Let A be a set of finite measure. Then there is a measure μ such that for each n, $mT^{-n}A = \int_I t^n\,d\mu$. It follows that

$$\begin{aligned}\int_A L(p)\,dm &= \sum a_n \int_A h_n\,dm = \sum a_n mT^{-n}A \\ &= \sum a_n \int_I t^n\,d\mu = \int_I p(t)\,d\mu \geq 0.\end{aligned}$$

Since A was chosen arbitrarily, $L(p) \geq 0$ a.e. Thus by Theorem 2, C is subnormal.

4. COROLLARY. *C is subnormal if and only if $\{h_n(x)\}$ is a moment sequence for almost every x in X.*

PROOF. Suppose that C is subnormal. Let $P_0(I)$ be the set of all polynomials over I with complex-rational coefficients, and let $P_0^+(I)$ be the set of polynomials in $P_0(I)$ which are nonnegative on I. For each p in $P_0^+(I)$, $L(p) \geq 0$ a.e. Let $X_p = \{x \text{ in } X\colon L(p)(x) \geq 0\}$, and let $X^* = \bigcup\{X_p\colon p \in P_0^+(I)\}$. Since $P_0^+(I)$ is countable and $m(X - X_p) = 0$ for each p in $P_0^+(I)$, $m(X - X^*) = 0$. Now it is easily verified that $P_0^+(I)$ is dense in the set of nonnegative polynomials over I. It follows that for every point x in X^* and every p in $P^+(I)$ we have $L(p)(x) \geq 0$. From the proposition listed above, $\{h_n(x)\}$ is a moment sequence for every x in X^*.

For the converse argument, suppose that $\{h_n(x)\}$ is a moment sequence for each x in Y, where Y is measurable and $m(X - Y) = 0$. If $p \in P^+(I)$, then by the proposition above, $L(p)(x) \geq 0$ on Y and thus by Theorem 2, since L is positive, C is subnormal.

Corollary 4 is particularly well suited for application to atomic measure spaces. For example suppose $X = \{0, 1, 2, \dots\}$ and m is given by $m(k) = m_k$, $k \in X$. Suppose $T(0) = 0$ and $T(k) = k-1$ for $k \geq 1$. We see that $h(0) = m(T^{-1}0)/m(0) = (m_0 + m_1)/m_0$; and for $k \geq 1$, $h(k) = m_{k+1}/m_k$. In general,

$$h_n(0) = (m_0 + \cdots + m_n)/m_0$$

and

$$h_h(k) = m_{k+n}/m_k \qquad (k \geq 1).$$

Without loss of generality assume that $m_0 = 1$. From Corollary 4 we see that C is subnormal if and only if for every $k \geq 0$ the sequences $\{\sum_{i=1}^n m_i\}$ and $\{m_{k+n}/m_k\}$ are moment sequences. For example, let $m_0 = 1$, $m_n = \int_1^2 (t^n - t^{n-1})\,dt$. Then $\sum_{i=0}^n m_i = \int_1^2 t^n\,dt = (2^{n+1} - 1)/(n+1)$. Now, for $k \geq 1$ and any $n \geq 0$,

$$\begin{aligned} \frac{m_{k+n}}{m_k} &= \frac{1}{m_k}\int_1^2 (t^{k+n} - t^{k+n-1})\,dt \\ &= \frac{1}{m_k}\int_1^2 t^n(t^k - t^{k-1})\,dt \\ &= \int_1^2 t^n\,d\mu_k, \end{aligned}$$

where $d\mu_k = (t^k - t^{k-1})\,dt/m_k$ defines a (nonnegative) measure on $[1, 2]$. Thus C is subnormal.

Another application of Corollary 4 is that of bilateral shifts on $l^2(-\infty, \infty)$. Suppose $\{m_k\colon -\infty < k < \infty\}$ is a two-sided sequence of positive numbers and T is given by $T(k) = k-1$. One easily verifies that for all $n \geq 0$ and all k, $h_n(k) = m_{n+k}/m_k$. It then follows readily that C is subnormal if and only if the two-sided sequence $\{m_n\}$ is a moment sequence. (Such a sequence is $m_n = \int_0^1 t^n e^{-t}\,dt$.) This result is precisely the characterization of subnormal bilateral shifts given in [**6**, p. 87].

REMARKS. 1. The preceding example was based on an invertible transformation T. It seems plausible that a characterization of subnormality for C where T is invertible can be found which is more tractable than those given in this paper. The situation in this case is reminiscent of that for weighted shifts, since if T is invertible then $h_n = h \cdot h \circ T^{-1} \cdots h \circ T^{-(n-1)}$.

2. In [**1**] it is shown that if $h \geq h \circ T$, then C^n is hyponormal for every $n \geq 0$. Not all such C are subnormal.

REFERENCES

1. P. Dibrell and J. Campbell, *Hyponormal powers of composition operators*, Proc. Amer. Math. Soc. **102** (1988), 914–918.
2. D. Harrington and R. Whitley, *Seminormal composition operators*, J. Operator Theory **11** (1984), 125–135.
3. A. Lambert, *Subnormality and weighted shifts*, J. London Math. Soc. (2) **14** (1976), 476–480.
4. ——, *Hyponormal composition operators*, Bull. London Math. Soc. **18** (1986), 395–400.
5. E. Nordgren, *Composition operators in Hilbert space*, Hilbert Space Operators, Lecture Notes in Math., vol. 639, Springer-Verlag, Berlin and New York, 1978.
6. A Shields, *Weighted shift operators and analytic function theory*, Topics in Operator Theory, Math. Surveys, no. 13, Amer. Math. Soc., Providence, R. I., 1974.
7. R. Singh and A. Kumar, *Characterization of invertible, unitary, and normal composition operators*, Bull. Austral. Math. Soc. **19** (1978), 81–95.
8. R. Singh, A. Kumar, and D. Gupta, *Quasinormal composition operators on* l_p^2, Indian J. Pure Appl. Math. **11** (7) (1980), 904–907.
9. R. Whitley, *Normal and quasinormal composition operators*, Proc. Amer. Math. Soc. **76** (1978), 114–118.
10. D. Widder, *The Laplace transform*, Princeton Univ. Press, Princeton, N. J., 1946.

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