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STRONG CONVERGENCE OF RESOLVENTS OF MONOTONE OPERATORS IN BANACH SPACES

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ABSTRACT. Let E^* be a real strictly convex dual Banach space with a Fréchet differentiable norm, and A a maximal monotone operator from E into E^* such that $A^{-1}0 \neq \emptyset$. Fix $x \in E$. Then $J_\lambda x$ converges strongly to Px as $\lambda \to \infty$, where J_λ is the resolvent of A, and P is the nearest point mapping from E onto $A^{-1}0$.

1. Introduction. Let E be a real Banach space, I the identity, and J the (normalized) duality mapping from E into E^*. Let B be an m-accretive operator in E such that $B^{-1}0 \neq \emptyset$. Then Reich [**10**] proved that, for every $x \in E$, $J_\lambda x = (I + \lambda B)^{-1}x$ converges strongly to Qx as $\lambda \to \infty$ when E is uniformly smooth, where Q is the unique sunny and nonexpansive retraction from E onto $B^{-1}0$. This theorem is useful to obtain strong convergence results for several explicit and implicit iteration methods for accretive operators, see [**10**].

The purpose of this paper is to obtain the analogous result for a maximal monotone operator A from E into E^*, which will be crucial to study iterations for monotone operators in Banach spaces. Suppose that $A^{-1}0 \neq \emptyset$. We know that, for every $z \in E^*$, $(J + \lambda A)^{-1}z$ converges strongly to Rz as $\lambda \to \infty$ when E^* is strictly convex and has a Fréchet differentiable norm, where Rz is the unique element of $A^{-1}0$ satisfying

$$\langle z - J(Rz), Rz - y\rangle \geq 0 \quad \text{for every } y \in A^{-1}0,$$

see Reich [**9**] and also [**3, 4**]. In this paper we study another convergence theorem to an element of $A^{-1}0$. Under some conditions, resolvents $J_\lambda : E \to E$, $\lambda > 0$, are defined for A, see §2. Then we prove that, for every $x \in E$, $J_\lambda x$ converges strongly to Px as $\lambda \to \infty$ when E^* has a Fréchet differentiable norm, where P is the unique nearest point retraction from E onto $A^{-1}0$. The contrast of these results becomes more striking when we characterize retractions, P and Q, analytically. That is,

$$Qx \text{ satisfies } \langle x - Qx, J(Qx - y)\rangle \geq 0 \quad \text{for all } y \in B^{-1}0,$$

and

> Px satisfies that for every $y \in A^{-1}0$, there is $z \in J(x - Px)$ such that $\langle z, Px - y\rangle \geq 0$,

see [**5, 7**], and also [**8**] for extensive study concerning such retractions.

Finally, let us consider briefly finding a sequence converging to a zero of the maximal monotone operator A. Fix an initial value x in E. Then, using the above

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result, $J_\lambda x$ approximates $Px \in A^{-1}0$ well for a sufficiently large λ (and for varying initial values, we obtain several elements of $A^{-1}0$). On the other hand, under some conditions, we obtain a sequence converging to $J_\lambda x$ by a gradient method, see Theorem 2 and Remark 5. Then, this sequence will be a good approximation to Px if λ is sufficiently large. It is an open problem to approximate zeros of monotone operators in Banach spaces by doubly iterations.

2. Main results. Let E^* be a real strictly convex dual Banach space with a Fréchet differentiable norm, and J be the (normalized) duality mapping from E into E^*, i.e., $Jx = \{y \in E^*: \langle x, y\rangle = \|x\|^2 = \|y\|^2\}$ for $x \in E$. Let A be a (multivalued) maximal monotone operator from E into E^*, i.e., $\langle y_1 - y_2, x_1 - x_2\rangle \geq 0$ for all $y_1 \in Ax_1$, $y_2 \in Ax_2$: $x_1, x_2 \in D(A)$, and A has no monotone extension. Fix $x \in E$. Then for every $\lambda > 0$ there exists a unique $x_\lambda \in D(A)$ such that $0 \in J(x_\lambda - x) + \lambda A x_\lambda$ (see [**1**, p. 104]). Putting $J_\lambda x = x_\lambda$, we define the *resolvent* $J_\lambda: E \to E$ of A for every $\lambda > 0$. Next, since A is a maximal monotone, $A^{-1}0$ is closed convex. If $A^{-1}0 \neq \varnothing$ then the strict convexity of E ensures the unique existence of the nearest point retraction P of E onto $A^{-1}0$. Then we prove

THEOREM 1. *Let E^* be a real strictly convex dual Banach space with a Fréchet differentiable norm, and A a maximal monotone operator from E into E^* such that $A^{-1}0 \neq \varnothing$. Then, for every $x \in E$, $J_\lambda x$ converges strongly to Px as $\lambda \to \infty$.*

PROOF. Fix $\lambda > 0$ arbitrarily. Then from the definition of x_λ $(= J_\lambda x)$ there exists $y_\lambda \in E^*$ such that y_λ belongs to both $J(x - x_\lambda)$ and $\lambda A x_\lambda$. For every $v \in A^{-1}0$, since A is monotone, we have

$$0 \leq \langle y_\lambda, x_\lambda - v\rangle = \langle y_\lambda, (x_\lambda - x) + (x - v)\rangle \leq -\|x_\lambda - x\|^2 + \|x_\lambda - x\| \cdot \|x - v\|.$$

Therefore we obtain

$$\|x_\lambda - x\| \leq \|x - v\| \quad \text{for all } v \in A^{-1}0 \text{ and } \lambda > 0. \tag{1}$$

Next, we show the weak convergence of x_λ to Px. By the inequality (1), we have $\|y_\lambda\|/\lambda = \|x - x_\lambda\|/\lambda \leq \|x - v\|/\lambda \to 0$ as $\lambda \to \infty$. Since E is reflexive we can take a subnet $\{x_{\lambda_\alpha}\}$ of $\{x_\lambda\}$ such that x_{λ_α} converges weakly to some $\bar{x} \in E$. Then since $(x_{\lambda_\alpha}, y_{\lambda_\alpha}/\lambda_\alpha) \in A$ and A is maximal monotone, $\bar{x} \in A^{-1}0$. Therefore by using (1) and the weak convergence of $x - x_{\lambda_\alpha}$ to $x - \bar{x}$, we have

$$\|x - \bar{x}\| \leq \liminf_{\alpha} \|x - x_{\lambda_\alpha}\| \leq \|x - v\| \quad \text{for all } v \in A^{-1}0. \tag{2}$$

Thus $\bar{x} = Px$. Since every convergent subnet has a unique convergent element Px, x_λ itself converges weakly to Px as $\lambda \to \infty$.

Then we obtain, as (2),

$$\|x - Px\| \leq \liminf_{\lambda} \|x - x_\lambda\| \leq \limsup_{\lambda} \|x - x_\lambda\| \leq \|x - Px\|.$$

That is $\|x - x_\lambda\|$ converges to $\|x - Px\|$ as $\lambda \to \infty$. Since E^* has a Fréchet differentiable norm, this implies the strong convergence of $x - x_\lambda$ to $x - Px$. Equivalently we obtain $x_\lambda \to Px$ as $\lambda \to \infty$.

REMARK 1. Instead of the normalized duality mapping J, the analogous result holds for the duality mapping J_ϕ with a gauge function ϕ.

REMARK 2. Fix $x \in E$. Instead of the exact form of $J_\lambda x$, let $x_\lambda \in E$, $\lambda > 0$, be a unique element satisfying $\varepsilon_\lambda \in J(x_\lambda - x) + \lambda A x_\lambda$ in E^*. If ε_λ converges to 0 as $\lambda \to \infty$ in E^*, then the same result as in Theorem 1 follows, i.e., $x_\lambda \to Px$ as $\lambda \to \infty$.

REMARK 3. From Theorem 1 and the proof of it, we have $A^{-1}0 = \varnothing$ if and only if $\lim_{\lambda\to\infty} \|J_\lambda x\| = \infty$.

REMARK 4. In the definition of $J_\lambda x$, the strict convexity of E^* is needed only to assert the existence of $J_\lambda x$ by using Corollary 4.1 of [**1**]. Therefore it is dropped when $R(J(\cdot - x) + \lambda A(\cdot)) = E^*$ is claimed by another reason. We say a monotone operator A from E into E^* satisfying such a condition is an *m-monotone* operator (with respect to J). When E^* is a strictly convex Banach space with a Fréchet differentiable norm, a maximal monotone operator from E into E^* is m-monotone. Another example of an m-monotone operator is the subdifferential of a lower-semicontinuous, proper and convex function on a reflexive Banach space. Then Theorem 1 holds if E^* has a Fréchet differentiable norm, and if A is a (multivalued) m-monotone operator from E into E^* such that $A^{-1}0 \neq \varnothing$.

Finally, we show a theorem to obtain the resolvent.

THEOREM 2. *Let E^* be a real dual Banach space with a Fréchet differentiable norm, J the (normalized) duality mapping from E into E^*, and A an m-monotone operator from E into E^*. Fix $x \in E$ and $\lambda > 0$. Define a monotone operator B from E into E^* by $B(y) = J(y-x) + \lambda A(y)$, $y \in D(A)$. Then if $\{(x_n, y_n)\}$ is a sequence in the graph of B such that $\{x_n\}$ is bounded and $y_n \to 0$ as $n \to \infty$, then x_n converges strongly to $J_\lambda x$ as $n \to \infty$.*

PROOF. Let $y_n = p_n + q_n, p_n \in J(x_n - x)$, $q_n \in \lambda A x_n$, and $r \in J(J_\lambda x - x) \cap -\lambda A(J_\lambda x)$. Then we obtain

$$\begin{aligned} \langle y_n, x_n - J_\lambda x\rangle &= \langle p_n + q_n, x_n - J_\lambda x\rangle \\ &= \langle p_n - r, x_n - J_\lambda x\rangle + \langle q_n + r, x_n - J_\lambda x\rangle \\ &\geq \langle p_n - r, x_n - J_\lambda x\rangle. \end{aligned}$$

Since $\{x_n\}$ is bounded and y_n converges strongly to 0, the left-hand side of the above inequality tends to 0 as $n \to \infty$. Therefore $\lim_n \langle p_n - r, (x_n - x) - (J_\lambda x - x)\rangle = 0$. Remark that $p_n \in J(x_n - x)$ and $r \in J(J_\lambda x - x)$. Since E^* has a Fréchet differentiable norm, this implies that $x_n - x$ converges strongly to $J_\lambda x - x$ as $n \to \infty$, equivalently x_n converges strongly to $J_\lambda x$ as $n \to \infty$.

REMARK 5. When A is the subdifferential of a lower-semicontinuous, proper and convex function f on E, then B is the subdifferential of $g(y) = \|y - x\|^2/2 + \lambda f(y)$, $y \in D(f)$. Then, under some additional assumptions, a sequence $\{x_n\}$ satisfying the whole condition of Theorem 2 is obtained by a gradient method for g, see [**2**].

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