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A QUESTION ON INVARIANT SUBSPACES OF BERGMAN SPACES

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ABSTRACT. We answer a question raised by Axler and Bourdon [1] concerning the classification of finite-codimensional invariant subspaces of Bergman spaces.

Let Ω be a bounded domain in $\mathbf{C}^n$, $1 \leq p < \infty$, and denote by $A^p(\Omega)$ the Bergman space consisting of all holomorphic functions in Ω, whose pth power has finite Lebesgue integral. A subspace $M \subset A^p(\Omega)$ is said to be invariant if $T_j M \subset M$, $1 \leq j \leq n$, where T_j denotes multiplication by the jth coordinate function z_j. Axler and Bourdon give in [1] a characterization of the finite-codimensional invariant subspaces of $A^p(\Omega)$ for sufficiently nice domains Ω. More precisely, they show that given such a subspace M, there exist linear partial differential operators $L_1, L_2, \ldots, L_q$ with constant coefficients, and points $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(q)} \in \Omega$, such that

$$M = \{f \in A^p(\Omega)\colon (L_j f)(\lambda^{(j)}) = 0,\ j = 1, 2, \ldots, q\}. \tag{1}$$

They raised the following natural question. Given linear partial differential operators $L_1, L_2, \ldots, L_q$ with constant coefficients, and points $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(q)} \in \Omega$, under what conditions is the space M defined by (1) invariant? Our purpose in this note is to provide such necessary and sufficient conditions. To formulate our result we need some notation. For a polynomial $P(\xi_1, \xi_2, \ldots, \xi_n) \in \mathbf{C}[\xi_1, \xi_2, \ldots, \xi_n]$, we denote by $P(D)$ the corresponding linear partial differential operator. Thus, if $P(\xi) = \xi_j$, then $P(D) = \partial/\partial z_j$, $1 \leq j \leq n$. Given $P \in \mathbf{C}[\xi_1, \xi_2, \ldots, \xi_n]$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Omega$, we define a functional $\varphi(P, \lambda)$ in the dual $(A^p(\Omega))'$ of $A^p(\Omega)$ by

$$\varphi(P,\lambda)(f) = (P(D)f)(\lambda), \qquad f \in A^p(\Omega).$$

Observe that the Leibniz formula (cf. formula (1.4.12) in [2]) yields immediately the equality

$$P(D)(T_j f) = T_j P(D) + \frac{\partial P}{\partial \xi_j}(D) f, \qquad 1 \leq j \leq n,$$

and hence

$$T_j' \varphi(P,\lambda) = \lambda_j \varphi(P,\lambda) + \varphi\left(\frac{\partial P}{\partial \xi_j}, \lambda\right), \qquad 1 \leq j \leq n, \tag{2}$$

where T_j' denotes the dual operator of T_j, acting on $(A^p(\Omega))'$.

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PROPOSITION. *Let $\Lambda \subset \Omega$ be a finite set, and for each $\lambda \in \Lambda$ let $\mathscr{F}_\lambda$ be a finite subset of $\mathbf{C}[\xi_1, \xi_2, \ldots, \xi_n]$. The subspace $M = \{f \in A^p(\Omega)\colon (P(D)f)(\lambda) = 0, P \in \mathscr{F}_\lambda, \lambda \in \Lambda\}$ is invariant if and only if for each $\lambda \in \Lambda$, the linear space generated by $\mathscr{F}_\lambda$ is invariant under $\partial/\partial\xi_j$, $1 \le j \le n$.*

PROOF. The space $M^\perp = \{\varphi \in (A^p(\Omega))'\colon \varphi|M = 0\}$ coincides with the linear space generated by $\{\varphi(P,\lambda)\colon \lambda \in \Lambda, P \in \mathscr{F}_\lambda\}$. Assume first that the linear space generated by $\mathscr{F}_\lambda$ is invariant under $\partial/\partial\xi_j$ for all $\lambda \in \Lambda$. Formula (2) shows then that $M^\perp$ is invariant under T_j', and hence M is invariant under T_j, $1 \le j \le n$. Conversely, assume that M is invariant under T_j. Then $M^\perp$ is invariant under T_j', $1 \le j \le n$. Again, formula (2) shows that in this case $\varphi(\partial P/\partial\xi_j, \lambda)$ must belong to $M^\perp$ whenever $\lambda \in \Lambda$, $P \in \mathscr{F}_\lambda$, and $1 \le j \le n$. Thus, for fixed λ, P and j, there exist scalars $\alpha_{\mu,Q}$ such that

$$\varphi\left(\frac{\partial P}{\partial \xi_j}, \lambda\right) = \sum_{\mu\in\Lambda}\sum_{Q\in\mathscr{F}_\mu} \alpha_{\mu,Q}\varphi(Q,\mu).$$

Since there are functions in $A^p(\Omega)$ that have zeros of arbitrarily high order at all points $\mu \in \Lambda\backslash\{\lambda\}$, and with prescribed partial derivatives up to a certain order at λ, the last equality implies that $\partial P/\partial\xi_j = \sum_{Q\in\mathscr{F}_\lambda} \alpha_{\lambda,Q}Q$. Thus $\partial P/\partial\xi_j$ belongs to the linear space generated by $\mathscr{F}_\lambda$. Since j, P, and λ are arbitrary, the proposition follows.

The criterion given in the above proposition is quite easy to apply. The reader will be able to verify at a glance that, for fixed $\lambda \in \Omega$ and $c \in \mathbf{C}$, the finite codimensional subspace

$$\left\{f \in A^p(\Omega)\colon f(\lambda) = \frac{\partial f}{\partial z_1}(\lambda) + \frac{\partial f}{\partial z_2}(\lambda) = c\frac{\partial^2 f}{\partial z_1|\partial z_2}(\lambda) + \frac{\partial^2 f}{\partial z_1^2}(\lambda) + \frac{\partial^2 f}{\partial z_2^2}(\lambda) = 0\right\}$$

is invariant if and only if $c = 2$. This example was suggested by the referee.

REFERENCES

1. S. Axler and P. Bourdon, *Finite codimensional invariant subspaces of Bergman spaces*, preprint.
2. L. Hörmander, *Linear partial differential operators*, Springer-Verlag, Berlin and New York, 1963.

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