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IDEMPOTENTS IN THE REDUCED C^*-ALGEBRA OF A FREE GROUP

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ABSTRACT. The purpose of this note is to give a simple proof of the theorem, originally due to M. Pimsner and D. Voiculescu, that the reduced C^*-algebra of a free group has no nontrivial idempotents.

1. Introduction. Let F be a free group. An element of F is a word in the generators and their inverses. We use $|x|$ for the length of the word $x \in F$. The group algebra $\mathbf{C}F$ consists of finite sums $\alpha = \sum \alpha_i g_i$, where $\alpha_i \in \mathbf{C}$ and $g_i \in F$. There are several extensions of $\mathbf{C}F$ which will be necessary for our study: define $|\alpha|_1 = \sum |\alpha_i|$ and $|\alpha|_2^2 = \sum |\alpha_i|^2$, and then let L^1F and L^2F be the completions of $\mathbf{C}F$ with respect to these norms. L^2F is a Hilbert space with Hilbert basis F and inner product $\langle\ ,\ \rangle$. Notice that by extending the multiplication on $\mathbf{C}F$ we can define a multiplication on L^1F and an action of L^1F on L^2F. The elements of L^1F act as bounded operators: since each element of F is a unitary operator on L^2F, the operator norm of any $\alpha \in L^1F$ satisfies $|\alpha|_1 \geq \|\alpha\| \geq |\alpha|_2$. The reduced C^*-algebra C_r^*F is the closure of $\mathbf{C}F$ with respect to $\|\ \|$.

Many years ago, Irving Kaplansky had suggested that a simple C^*-algebra is generated by its projections (selfadjoint idempotents). Richard Kadison offered C_r^*F as a counterexample, conjecturing that it was both simple and without nontrivial idempotents. In [**P**] Robert Powers showed that C_r^*F was simple, leaving the interesting problem of showing that C_r^*F is without idempotents, or rather the only idempotents are 0 and 1. As a first step towards this, one of us proved in [**CH**] that L^1F has no idempotents. (It was actually shown there that the full C^*-algebra—which comes from considering all unitary representations at once—has no idempotents.) The full result, that is,

THEOREM. *C_r^*F is without nontrivial idempotents,*

was proved first by Pimsner and Voiculescu [**PV**], and later by many others: [**CZ**, **L**, **CN**, **JV**, **K**]. All of the proofs were either complicated in themselves, or not self-contained, in that they relied on very heavy and sophisticated machinery of K-theory.

Our purpose here is to present a proof sufficiently elementary that any reader who fully understands the question can follow the proof. We claim no originality for this proof. It is basically contained in [**CN**] where, however, it is shrouded in the

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terminology of noncommutative differential geometry. The fact that an elementary proof of this theorem could be written down on the basis of existing knowledge was probably clear to many experts. Lemma 4 was suggested to one of us by R. Szwarč, who apparently learned it from U. Haagerup.

Let $x \in F$ be any element except the identity. Let $\gamma(x)$ be the word obtained from x by removing the last letter in the word x. Let S be the Hilbert space whose basis elements consist of the symbol ν and the unordered pairs $\{x, y\}$ of elements of F such that $|x^{-1}y| = 1$; that is, x is obtained from y by adding or deleting the last letter on the right. Let F act on S as follows: for $g \in F$, $g\{x, y\} = \{gx, gy\}$ and $g\nu = 0$. There is an isomorphism of Hilbert spaces $P\colon L^2F \to S$ given by $P(x) = \{x, \gamma(x)\}$ for $x \neq 1$, and $P(1) = \nu$. Since P is a bijection of orthonormal bases, P^* is the inverse of P.

There is an action of F on L^2F by left multiplication. Note that P is almost, but not quite, F-equivariant: for $g \in F$ and $x \in F, x \neq 1$, $P^*gPx = P^*\{gx, g\gamma(x)\}$, which is the longer of the two words gx, $g\gamma(x)$. If $g = a_1a_2\cdots a_n$, then gx is always longer except in the case that $x^{-1} = a_ia_{i+1}\cdots a_n$ for some $i = 1, \ldots, n$; that is, except in the case that x^{-1} is the right of g. Let $\mathbf{E}(g)$ be the set of inverses of the elements $a_1\cdots a_n$, $a_2\cdots a_n, \ldots, a_n, 1$. Then $P^*gPx = gx$, except for $x \in \mathbf{E}(g)$. Setting $\gamma(1) = 0$, we can write

$$P^*gPx = \begin{cases} gx & \text{for } x \notin \mathbf{E}(g), \\ g\gamma(x) & \text{for } x \in \mathbf{E}(g). \end{cases}$$

In particular the operator $(g - P^*gP)$ has rank $|g| + 1$ acting on the space L^2F.

Extending this argument linearly to $\mathbf{C}F$, we find that for any $a \in \mathbf{C}F$, $(a - P^*aP)$ is a finite rank operator. Let $a \in C_r^*F$ be such that $a - P^*aP$ acts on L^2F as a trace class operator. This is true, for example, for $a \in \mathbf{C}F$. Let us calculate the diagonal terms of $a - P^*aP$ with respect to the orthonormal basis F: For $g, x \in F$,

$$\langle (g - P^*gP)x, x\rangle = \begin{cases} 0, & x \notin \mathbf{E}(g), \\ \langle gx - g\gamma(x), x\rangle, & x \in \mathbf{E}(g). \end{cases}$$

For $x \neq 1$, this is necessarily 0, and for $x = 1$ this is $\langle g, 1\rangle$. Thus $\langle (a - P^*aP)x, x\rangle = 0$ for $x \neq 1$ and for $x = 1$ is $\langle a, 1\rangle$. So $\operatorname{tr}(a - P^*aP)$ is $\langle a, 1\rangle$, exactly the coefficient of the identity in a, which we denote by $\tau(a)$. We refer to $\tau(a)$ as the τ-trace of a.

Let us set $\mathscr{A} = \{a \in C_r^*F \mid (a - P^*aP) \text{ is of trace class}\}$. Notice that $\mathbf{C}F \subseteq \mathscr{A}$. Then for all $a \in \mathscr{A}$, $\tau(a) = \operatorname{tr}(a - P^*aP)$.

We now give the proof of the main result using the lemmas which are proved in the next section: Assume that we are given a nontrivial idempotent $f \in C_r^*F$ (i.e. $f \neq 0$ or 1). Then Lemma 1, due to Kaplansky, shows that there is a projection (that is, a selfadjoint idempotent) $e \in C_r^*F$, which is also nontrivial. We then show (Lemma 3) that given the projection e, we can find a nontrivial projection $p \in \mathscr{A}$. Notice that p a projection implies that P^*pP is also a projection. Thus if p is a projection in $\mathscr{A}$, then Lemma 4 proves that $\operatorname{tr}(p - P^*pP)$ is necessarily an integer. Finally, another trick due to Kaplansky (Lemma 5) shows that a nontrivial idempotent cannot have integral τ-trace. This contradicts the fact that p is nontrivial, and that $\tau(p) = \operatorname{tr}(p - P^*pP)$, and proves the main result. □

2. Statements and proofs of lemmas.

LEMMA 1. *If f is an idempotent in a $*$-ring and $z = 1 + (f - f^*)(f^* - f)$ has an inverse t, then $e = ff^*t$ is a projection and $fe = e$, $ef = f$. In particular, then,* $\operatorname{trace}(e) = \operatorname{trace}(f)$ *for any type of trace that makes sense.*

PROOF. Observe that f and f^* commute with t since they commute with z and that t is selfadjoint. Thus e is selfadjoint. Also note that $fz = ff^*f$, so that $f = ff^*ft = ff^*tf = ef$. It follows from the definition that $fe = e$, and thus $e^2 = efe = fe = e$. □

LEMMA 2. *If $a \in \mathscr{A}$ and $\phi_\lambda = (\lambda - a)^{-1} \in C_r^*F$, then $\phi_\lambda \in \mathscr{A}$.*

PROOF. $a\phi_\lambda = \lambda\phi_\lambda - I$, by definition. Thus

$$\begin{aligned}\phi_\lambda - P^*\phi_\lambda P &= P^*\phi_\lambda P\lambda\phi_\lambda - P^*\phi_\lambda P - P^*\phi_\lambda P\lambda\phi_\lambda + \phi_\lambda \\ &= P^*\phi_\lambda Pa\phi_\lambda - P^*\phi_\lambda aP\phi_\lambda = P^*\phi_\lambda P(a - P^*aP)\phi_\lambda.\end{aligned}$$

Since operators of trace class form a two-sided ideal, $\phi_\lambda - P^*\phi_\lambda P$ is of trace class and $\phi_\lambda \in \mathscr{A}$. Notice also that $\phi_\lambda - P^*\phi_\lambda P$ is continuous in the trace norm as a function of λ. □

Observe that Lemma 1 applies to C_r^*F and the τ-trace, and that Lemma 2 implies that the spectrum of a in C^*F is the same as its spectrum in $\mathscr{A}$.

LEMMA 3. *If there is a nontrivial projection in C_r^*F, then there is a nontrivial projection in $\mathscr{A}$.*

PROOF. Let $e \in C_r^*F$ be a nontrivial projection. Then the spectrum of e in C_r^*F is $\{0, 1\}$. Now $\mathscr{A}$ is dense in C_r^*F (since $\mathbf{C}F$ is) and so we can find $q \in \mathscr{A}$ selfadjoint and arbitrarily close to e, in particular such that the spectrum of q is disconnected and real. Notice that $\mathscr{A}$ is closed under integration along a closed interval: If $q(t)$ is a continuous family of elements of $\mathscr{A}$ for $0 \le t \le 1$, and $Q = \int q(t)dt$, then $Q - P^*QP = \int (q(t) - P^*q(t)P)dt$, a continuous integral of elements of trace class, is of trace class, whence $Q \in \mathscr{A}$. Now construct a nontrivial projection in $\mathscr{A}$ as follows: Let ϕ_λ be $[1/2\pi i](\lambda - q)^{-1}$ for λ not in the spectrum of q. By Lemma 2, $\phi_\lambda \in \mathscr{A}$. We can integrate ϕ_λ around a simple closed curve that contains one but not all of the components of the spectrum of q. By the preceding remarks, the resulting element, p, a projection necessarily different from 0 and 1, is in $\mathscr{A}$. □

LEMMA 4. *If P and Q are projections and $P-Q$ is of trace class, then* $\operatorname{tr}(P-Q)$ *is an integer.*

PROOF. Note that $P(P - Q)^2 = P - PQP = (P - Q)^2P$. Since $P - Q$ is of a trace class, it is compact and we can thus write it as $\sum \lambda E_\lambda$ where E_λ is the projection onto U_λ, the finite-dimensional λ-eigenspace of $P-Q$. Let $v \in U_\mu$. Then

$$\sum \lambda^2 E_\lambda Pv = (P - Q)^2 Pv = P(P - Q)^2 v = P\mu^2 v = \mu^2 Pv.$$

So Pv is in the μ^2-eigenspace of $(P - Q)^2$, which is $W_\mu = U_\mu \oplus U_{-\mu}$. Thus $P(W_\mu) \subseteq W_\mu$. Similarly $Q(W_\mu) \subseteq W_\mu$. Now $P \,|\, W_\mu$ and $Q \,|\, W_\mu$ are projections on a finite dimensional space, so both have integral trace. Thus $\operatorname{tr}(P - Q) \,|\, W_\mu$ is an integer. Since the support of $P - Q$ is $\bigoplus W_\mu$ and $P - Q$ is of trace class, this shows that $\operatorname{tr}(P - Q)$ is an integer. □

The final step is the following simple observation of Kaplansky:

LEMMA 5. *If $P \in C_\tau^* F$ is a projection, then $0 \leq \tau(P) \leq 1$ and $\tau(P) = 0$ or 1 if and only if $P = 0$ or 1.*

PROOF. $\tau(P) = \langle P, 1\rangle = \langle P^2, 1\rangle = \langle P, P^*\rangle = \langle P, P\rangle = |P|_2^2$ which is ≥ 0 and $= 0$ if and only if $P = 0$. But $1 - P$ is also a projection so $1 - \tau(P) = \tau(1 - P) \geq 0$ and $= 0$ if and only if $1 - P = 0$. □

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