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KARP'S THEOREM IN ACOUSTIC SCATTERING THEORY

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ABSTRACT. Karp's Theorem states that if the far field pattern corresponding to the scattering of a time harmonic plane acoustic wave by a sound-soft cylinder is of the form $F_0(k; \theta - \alpha)$ where k is the wave number, θ the angle of observation and α the angle of incidence of the plane wave, then the cylinder must be circular. A new proof is given of this result and extended to the cases of scattering by a sound-hard obstacle and an inhomogeneous medium.

1. Introduction. Consider the problem of the scattering of a time harmonic plane acoustic wave by a sound-soft cylinder of cross section D. If we assume that the z axis coincides with the axis of the cylinder and the incident plane wave propagates in a direction that is perpendicular to the cylinder, then, factoring out the time harmonic component, the velocity potential u of the scattered wave is a function only of the Cartesian coordinates x and y and has the asymptotic behavior

$$u(r, \theta) = \frac{e^{ikr}}{\sqrt{r}} F(k; \theta, \alpha) + O(r^{-3/2}) \tag{1.1}$$

where (r, θ) are the polar coordinates of (x, y), $k > 0$ is the wave number, and α is the angle of incidence of the plane wave. The *inverse scattering problem* is to determine D from a knowledge of the *far field pattern* F. To date, there is only one explicitly solvable problem in inverse scattering theory and the solution is known as Karp's Theorem: If D is sound-soft and $F(k; \theta, \alpha)$ is of the form

$$F(k; \theta, \alpha) = F_0(k; \theta - \alpha) \tag{1.2}$$

for some function F_0, then D is a disk [**8**, **11**]. The aim of this note is to extend Karp's Theorem to include the cases of a sound-hard scattering obstacle and scattering by an inhomogeneous medium. Since the proof of Karp's Theorem cannot be modified to include these two cases, we shall present a totally new approach to the problem. For purposes of exposition and completeness we shall begin below by using our methods to reprove Karp's Theorem for a sound-soft obstacle.

As pointed out by Sleeman in [**11**], the importance of an explicitly solvable problem in inverse scattering theory is that the solution suggests methods for solving the inverse scattering problem for arbitrary obstacles (or inhomogeneities). In particular, the method of Karp is based on analytic continuation and is intimately connected with the Imbriale-Mittra method for solving the inverse scattering problem for sound-soft cylinders of arbitrary cross section [**7**]. In contrast, our method

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is based on the evaluation for the scattering matrix applied to specific functions in $L^2[-\pi,\pi]$ and in this sense is closely related to the projection methods recently developed by Colton and Monk to solve the inverse scattering problem for arbitrary obstacles and inhomogeneities (cf. [**2**, **3**]).

For the sake of simplicity, we have only considered two dimensional scattering problems. However, all of our results can easily be extended to the three dimensional case where it is now assumed that the far field pattern is of the form

$$F(k;\hat{\mathbf{x}},\boldsymbol{\alpha}) = F_0(k;\hat{\mathbf{x}}\cdot\boldsymbol{\alpha}) \tag{1.3}$$

where $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$, $\mathbf{x}\in R^3$, and $\boldsymbol{\alpha}$, $|\boldsymbol{\alpha}|=1$, is the direction of propagation of the incident plane wave. In this case, use must be made of the Funk-Hencke Theorem for spherical harmonics [**6**, p. 247] and Gegenbauer's integral formula [**6**, p.178] at appropriate points in the proofs below.

II. The sound-soft obstacle. Let D be a bounded, simply connected domain containing the origin with C^2 boundary ∂D. Then, if D is sound-soft, the velocity potential u of the scattered field satisfies the equations (cf. [**1**])

$$\Delta_2 u + k^2 u = 0 \quad \text{in } R^2\backslash\overline{D}, \tag{2.1}$$

$$u(\mathbf{x}) = -\exp[ik\mathbf{x}\cdot\boldsymbol{\alpha}] \quad \text{on } \partial D, \tag{2.2}$$

$$\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial u}{\partial r} - iku\right) = 0, \tag{2.3}$$

where $\mathbf{x} = r(\cos\theta,\sin\theta)$, $\boldsymbol{\alpha} = (\cos\alpha,\sin\alpha)$, and $\exp[ik\mathbf{x}\cdot\boldsymbol{\alpha}]$ is the incident field. The function u has the asymptotic behavior given by (1.1). The following theorem (but not the proof) is due to Karp.

THEOREM 1. *Let D be sound-soft and suppose* (1.2) *is true for some fixed k and all $\alpha\in[-\pi,\pi]$, $\theta\in[-\pi,\pi]$. Then D is a disk.*

PROOF. From (1.2) we have that for all $\alpha\in[-\pi,\pi]$,

$$\int_{-\pi}^{\pi} F(k;\theta,\alpha)\,d\theta = \int_{-\pi}^{\pi} F_0(k;\theta-\alpha)\,d\theta = \int_{-\pi}^{\pi} F_0(k;\theta)\,d\theta. \tag{2.4}$$

Hence, if α_1 is a fixed angle then for all $\alpha\in[-\pi,\pi]$ we have

$$\int_{-\pi}^{\pi}[F(k;\theta,\alpha) - F(k;\theta,\alpha_1)]\,d\theta = 0. \tag{2.5}$$

This shows that $g(\theta)=1$ is orthogonal to

$$S = \operatorname{span}\{F(k;\theta,\alpha) - F(k;\theta,\alpha_1)\colon \alpha\in[-\pi,\pi]\} \tag{2.6}$$

in $L^2[-\pi,\pi]$. From [**9**] (see also [**3**]) we can now conclude that the Herglotz wave function

$$J_0(kr) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\exp[ikr\cos(\theta-\phi)]\,d\phi \tag{2.7}$$

coincides with $cH_0^{(2)}(kr)$ on ∂D for some constant c where J_0 denotes a Bessel function of order zero and $H_0^{(2)}$ a Hankel function of the second kind of order zero. Hence, there exist real constants c_1 and c_2, not both zero, such that

$$c_1 J_0(kr) + c_2 Y_0(kr) = 0 \tag{2.8}$$

for all $\mathbf{x} = r(\cos\theta, \sin\theta) \in \partial D$ where Y_0 denotes a Neumann function of order zero. Since J_0 and Y_0 are linearly independent, this implies that if $\mathbf{x} \in \partial D$ then $r = |\mathbf{x}|$ =constant, i.e. ∂D is a circle and hence D is a disk.

III. The sound-hard obstacle. Let D be as in §II and let $\nu = \nu(\mathbf{x})$ denote the unit outward normal to ∂D. Then, if D is sound-hard, the velocity potential u of the scattered field satisfies the equations (cf. [**1**])

$$\Delta_2 u + k^2 u = 0 \quad \text{in } R^2\backslash\overline{D}, \tag{3.1}$$

$$\frac{\partial}{\partial\nu}u(\mathbf{x}) = -\frac{\partial}{\partial\nu}\exp[ik\mathbf{x}\cdot\boldsymbol{\alpha}] \quad \text{on } \partial D, \tag{3.2}$$

$$\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial u}{\partial r} - iku\right) = 0, \tag{3.3}$$

where $\mathbf{x} = r(\cos\theta, \sin\theta)$, $\boldsymbol{\alpha} = (\cos\alpha, \sin\alpha)$. The function u again has the asymptotic behavior given by (1.1). We now have the following analogue of Karp's Theorem.

THEOREM 2. *Let D be sound-hard and suppose* (1.2) *is true for some fixed k and all $\alpha \in [-\pi, \pi]$, $\theta \in [-\pi, \pi]$. Then D is a disk.*

PROOF. As in Theorem 1, we can conclude that $g(\theta) = 1$ is orthogonal in $L^2[-\pi, \pi]$ to the set S defined by (2.6). From [**9**] we can now conclude that there exists a constant c such that

$$\frac{\partial}{\partial\nu}J_0(kr) = c\frac{\partial}{\partial\nu}H_0^{(2)}(kr) \tag{3.4}$$

for $\mathbf{x} = r(\cos\theta, \sin\theta) \in \partial D$. Hence, there exist real constants c_1 and c_2, not both zero, such that

$$c_1\frac{\partial}{\partial\nu}J_0(kr) + c_2\frac{\partial}{\partial\nu}Y_0(kr) = 0 \tag{3.5}$$

for $\mathbf{x} = r(\cos\theta, \sin\theta) \in \partial D$. Since $J_0'(z) = -J_1(z)$, $Y_0'(z) = -Y_1(z)$, where J_1 is a Bessel function of order one and Y_1 is a Neumann function of order one, we can rewrite (3.5) in the form

$$(c_1 J_1(kr) + c_2 Y_1(kr))\nu_r(\mathbf{x}) = 0 \tag{3.6}$$

for $\mathbf{x} = r(\cos\theta, \sin\theta) \in \partial D$ where ν_r denotes the radial component of the unit normal ν.

Now define the set A by

$$A = \{\mathbf{x} \in \partial D\colon \nu_r(\mathbf{x}) \neq 0\}. \tag{3.7}$$

Then A is open. We shall show that $\nu_r(\mathbf{x}) = 1$ for $\mathbf{x} \in A$. Assume, on the contrary, that $\nu_r(\mathbf{x}_0) \neq 1$ for some $\mathbf{x}_0 \in A$. Then there exists a curve $C \subset A$ such that $\nu_r(\mathbf{x}) \neq 1$ for all $\mathbf{x} \in C$. But then from (3.6) we have that

$$c_1 J_1(kr) + c_2 Y_1(kr) = 0 \tag{3.8}$$

for $\mathbf{x} = r(\cos\theta, \sin\theta) \in C$ and as in Theorem 1 we can conclude that C is an arc of a circle. Hence $\nu_r(\mathbf{x}) = 1$ for $\mathbf{x} \in C$, a contradiction.

We now see that for $\mathbf{x} \in \partial D$ the function $\nu_r(\mathbf{x})$ is either zero or one. Since $\nu_r(\mathbf{x})$ is continuous for $\mathbf{x} \in \partial D$ and ∂D is a closed curve, we have that $\nu_r(\mathbf{x}) = 1$ for all $x \in \partial D$. Hence, ∂D is a circle, i.e. D is a disk.

IV. The inhomogeneous medium. We now want to prove the analogue of Karp's Theorem for wave propagation in an inhomogeneous medium where the region of inhomogeneity has compact support. In particular, we want to show that if the far field pattern is of the form (1.2) then, under appropriate assumptions, the inhomogeneous medium must be spherically stratified. More specifically, assume that the density is constant but the compressibility $\kappa = \kappa(\mathbf{x})$ is space dependent and $\kappa(\mathbf{x}) = \kappa_0$ =constant for $|\mathbf{x}| \geq a > 0$. Then if the incident field is given by $\exp[ik\mathbf{x}\cdot\boldsymbol{\alpha}]$, the velocity potential $u = u(\mathbf{x};\boldsymbol{\alpha})$ of the total field satisfies the equations (cf. [**5**])

$$\Delta_2 u + k^2(1 - m(\mathbf{x}))u = 0 \quad \text{in } R^2, \tag{4.1}$$

$$u(\mathbf{x};\boldsymbol{\alpha}) = \exp[ik\mathbf{x}\cdot\boldsymbol{\alpha}] + u^s(\mathbf{x};\boldsymbol{\alpha}), \tag{4.2}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \tag{4.3}$$

where $\mathbf{x} = r(\cos\theta, \sin\theta)$, $\boldsymbol{\alpha} = (\cos\alpha, \sin\alpha)$,

$$m(\mathbf{x}) = \frac{\kappa_0 - \kappa(\mathbf{x})}{\kappa_0} \tag{4.4}$$

is continuously differentiable for $|\mathbf{x}| \leq a$ and $u^s = u^s(\mathbf{x};\boldsymbol{\alpha})$ denotes the velocity potential of the scattered field.

We can rewrite (4.1)–(4.3) as the integral equation

$$\begin{aligned}\exp[ik\mathbf{x}\cdot\boldsymbol{\alpha}] &= u(\mathbf{x};\boldsymbol{\alpha}) + \frac{ik^2}{4}\iint_B H_0^{(1)}(k|\mathbf{x}-\mathbf{y}|)m(\mathbf{y})u(\mathbf{y};\boldsymbol{\alpha})\,dy \\ &= (\mathbf{I} + k^2\mathbf{T})u\end{aligned} \tag{4.5}$$

where $H_0^{(1)}$ denotes a Hankel function of the first kind of order zero and

$$B = \{\mathbf{x}\colon |\mathbf{x}| < a\}. \tag{4.6}$$

Then from the asymptotic behavior of the Hankel function we see that u^s has the asymptotic behavior

$$u^s(\mathbf{x};\boldsymbol{\alpha}) = \frac{e^{ikr}}{\sqrt{r}}F(k;\theta,\alpha) + O(r^{-3/2}) \tag{4.7}$$

where the far field pattern F is given by

$$F(k;\theta,\alpha) = -e^{i\pi/4}\sqrt{\frac{k^3}{8\pi}}\iint_B m(\mathbf{y})u(\mathbf{y};\boldsymbol{\alpha})\exp[-ik\rho\cos(\theta-\phi)]\,d\mathbf{y} \tag{4.8}$$

and $\mathbf{y} = \rho(\cos\phi, \sin\phi)$. We can now state and prove the following analogue of Karp's Theorem.

THEOREM 3. *Let F be the far field pattern corresponding to* (4.1)–(4.3) *and suppose* (1.2) *is true for all $k > 0$, $\alpha \in [-\pi,\pi]$, $\theta \in [-\pi,\pi]$. Then $m(\mathbf{x})$ is spherically stratified, i.e. $m(\mathbf{x}) = m_0(r)$ for some function m_0.*

PROOF. Suppose (1.2) is true. Then from the Jacobi-Anger expansion

$$\exp[ikr\cos\theta] = \sum_{n=-\infty}^{\infty} i^n J_n(kr)e^{in\theta} \tag{4.9}$$

where J_n denotes a Bessel function of order n we have that

$$\int_{-\pi}^{\pi} F(k;\theta,\alpha)e^{in\theta}\,d\theta = e^{in\alpha}\int_{-\pi}^{\pi} F_0(k;\theta)e^{in\theta}\,d\theta \tag{4.10}$$
$$= -e^{i\pi/4}\sqrt{\frac{k^2}{8\pi}}(-i)^n \iint_B m(\mathbf{y})u(\mathbf{y};\boldsymbol{\alpha})J_n(k\rho)e^{in\phi}\,d\mathbf{y}.$$

But from (4.5) we have that

$$u(\mathbf{x};\boldsymbol{\alpha}) = (\mathbf{I}+k^2\mathbf{T})^{-1}(\exp[ik\mathbf{x}\cdot\boldsymbol{\alpha}]) \tag{4.11}$$

and hence for any integer q we have that

$$\int_{-\pi}^{\pi} u(\mathbf{x};\boldsymbol{\alpha})e^{iq\alpha}\,d\alpha = (\mathbf{I}+k^2\mathbf{T})^{-1}(i^q J_q(kr)e^{iq\theta}). \tag{4.12}$$

Noting that $\|\mathbf{T}\|_{L^2(B)} = O(|\log k|)$, we see that the Neumann series for $(\mathbf{I}+k^2\mathbf{T})^{-1}$ converges for k sufficiently small. Furthermore,

$$\lim_{k\to 0} k^{-n}J_n(kr) = \frac{r^n}{2^n n!}, \qquad n \geq 0, \tag{4.13}$$
$$J_{-n}(kr) = (-1)^n J_n(kr).$$

Hence, multiplying (4.10) by $e^{iq\alpha}$, integrating from $-\pi$ to π, dividing by k^{n+q}, and letting k tend to zero shows that for $n \neq -q$ we have that

$$\iint_B \rho^{|n|e^{in\phi}}\rho^{|q|}e^{iq\phi}m(\mathbf{y})\,d\mathbf{y} = 0, \qquad n \neq -q. \tag{4.14}$$

Now set

$$a_0(\rho) = \frac{1}{2\pi}\int_{-\pi}^{\pi} m(\rho,\phi)\,d\phi \tag{4.15}$$

where $m(\mathbf{y}) = m(\rho,\phi)$. Then from (4.14) we have that

$$\int_0^a \int_{-\pi}^{\pi} \rho^{|n|+|q|+1}e^{i(n+q)\phi}[m(\rho,\phi)-a_0(\rho)]\,d\phi\,d\rho = 0 \tag{4.16}$$

for all integers n and q. Since the set of products of harmonic polynomials is complete in $L^2(B)$ [**10**], we can now conclude that

$$m(r,\theta) = a_0(r), \tag{4.17}$$

i.e. $m(\mathbf{x})$ is spherically stratified. More directly, if

$$m(\rho,\phi) = \sum_{l=-\infty}^{\infty} a_l(\rho)e^{il\phi} \tag{4.18}$$

and n and q are such that $n+q=-l$ for some fixed integer $l \neq 0$, then from (4.16) we have that

$$\int_0^a \rho^{|n|+|l+n|+1}a_l(\rho)\,d\rho = 0 \tag{4.19}$$

for each $l \neq 0$ and all integers n. By Muntz's Theorem (cf. [**4**]) we can now conclude that $a_l(\rho) = 0$ for each $l \neq 0$ and hence from (4.18) $m(\rho,\phi) = a_0(\rho)$.

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