PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 103, Number 3, July 1988

CONSTRUCTION OF HAMILTON SEQUENCES FOR CERTAIN TEICHMÜLLER MAPPINGS

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(Communicated by Irwin Kra)

ABSTRACT. Suppose $\sup\{|\iint_{|z|<1} \kappa(z)\varphi(z)\,dx\,dy| \colon \varphi(z)$ analytic for $|z| < 1$, $\iint_{|z|<1} |\varphi(z)|\,dx\,dy = 1\} = \operatorname{ess\,sup}\{|\kappa(z)| \colon |z| < 1\}$. The question of constructive determination of extremal sequences $\{\varphi_n\}$ is considered for some classes of functions $\kappa(z)$ that arise in connection with plane quasiconformal mappings. For example, such a sequence $\{\varphi_n\}$ is constructed explicitly for the $\kappa(z)$ that arises in connection with the affine stretch of Strebel's chimney domain.

1. Introduction. f is a Teichmüller mapping of the unit disk, $U = \{|z| < 1\}$, if f is a quasiconformal mapping of U with complex dilatation

$$\frac{f_{\bar z}}{f_z} = k\frac{\overline{\phi(z)}}{|\phi(z)|}, \qquad z \in U, \tag{1.1}$$

where $\phi(z)$ is holomorphic in U, and k is a positive constant. Let $\mathfrak{B}(\Omega)$ denote the class of functions $\phi(z)$ holomorphic in a region Ω, with the additional restriction that

$$0 < \|\phi(z)\| = \iint_{\Omega} |\phi(z)|\,dx\,dy < \infty, \qquad z = x + iy.$$

A necessary and sufficient condition that f is an extremal mapping (among the class of quasiconformal mappings of U with the same boundary values as f) is that [**2**] there exists a so-called *Hamilton sequence*, namely, a sequence $\phi_n \in \mathfrak{B}(U)$, such that

$$\lim_{n\to\infty} \frac{\left|\iint_U (\overline{\phi(z)}/|\phi(z)|)\phi_n(z)\,dx\,dy\right|}{\|\phi_n(z)\|} = 1. \tag{1.2}$$

If $\phi \in \mathfrak{B}(U)$, then, of course, $\phi_n(z) = \phi(z)$, $n = 1, 2, 3, \ldots$, constitutes a Hamilton sequence; so the problem of whether a Hamilton sequence exists is nontrivial only when $\|\phi(z)\| = \infty$. In principle, if a Hamilton sequence, $\{\phi_n\}$, does exist, such a sequence can be realized in terms of a sequence of "polygonal" Teichmüller mappings f_n which agree with f at finitely many boundary points (see [**2**, Theorem 6]). However, these f_n's are obtained by a highly nonconstructive process, with the result that the relationship of the properties of the corresponding sequence $\{\phi_n\}$ to the properties of ϕ is quite obscure. The question arises whether a Hamilton sequence, if one does exist, can be obtained in a more direct manner. In particular, we shall concern ourselves with the following possibility: *If* $\{R_n\}$ *is a sequence of numbers,*

Received by the editors April 23, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 30C60, 30C75; Secondary 30H05.

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0002-9939/88 $1.00 + $.25 per page

$0 < R_n < 1$, $\lim R_n = 1$, *does* $\{\phi(R_n z)\}$ *constitute a Hamilton sequence*? We shall answer this question principally in the special case when $\phi(z)$ *is holomorphic in* $\overline{U}$ *except for a finite number of poles on* ∂U. In §2, we shall verify the following:

THEOREM 1. *Suppose* $\phi(z)$ *is holomorphic for* $|z| \leq 1$ *except for a finite number of poles on* $\{|z| = 1\}$. *Then*

$$\lim_{R\to 1} \frac{\left|\iint_U (\overline{\phi(z)}/|\phi(z)|)\phi(Rz)\,dx\,dy\right|}{\|\phi(Rz)\|} = 1 \tag{1.3}$$

if and only if $\phi(z)$ *has poles of at most order* 2 *on* $\{|z| = 1\}$.

As was first noted by Strebel [**5**], with his famous chimney region, an extremal Teichmüller mapping is not necessarily *uniquely* extremal. For the class of ϕ's under consideration here, Sethares [**4**] showed that unique extremality obtains if and only if the poles are of the first or second order. Thus, Theorem 1 is equivalent to the following.

THEOREM 1′. *Suppose* $\phi(z)$ *is holomorphic for* $|z| \leq 1$ *except for a finite number of poles on* $\{|z| = 1\}$. *Then* $\{\phi(R_n z)\}$ *is a Hamilton sequence if and only if the Teichmüller mapping* (1.1) *is uniquely extremal.*

The "if" part, above, is in fact known to hold for a somewhat larger class of ϕ's. Examination of the proof of Theorem 2 of Hayman and Reich [**1**] shows that what is proved there is actually tantamount to the following.

THEOREM 2. *Suppose* $\phi(z)$ *is holomorphic for* $|z| < 1$, *and*

$$\int_0^{2\pi} |\phi(re^{i\theta})|\,d\theta \leq \frac{1}{1-r}, \qquad 0 \leq r < 1.$$

Then the Teichmüller mapping (1.1) *is uniquely extremal and* $\{\phi(R_n z)\}$ *is a Hamilton sequence.*

The proof of Theorem 2 is complicated. Thus, even though Theorem 2 implies the "if" part of Theorem 1, we include a proof of the latter in §2 since only a short computation is involved.

In §3 we interpret our results for affine mappings of certain simply connected regions, and in §4 we consider the question of Hamilton sequences for the affine stretch of Strebel's chimney region.

2. Proof of Theorem 1. (a) Suppose $\|\phi(z)\| < \infty$ (as is the case if ϕ has at worst first order poles).

Given $\varepsilon > 0$, choose R_0 such that

$$\iint_{R_0^2<|z|<1} |\phi(z)|\,dx\,dy < \varepsilon.$$

Then,

$$\iint_{R_0<|z|<1} |\phi(Rz)|\,dx\,dy = \frac{1}{R^2}\iint_{R_0 R<|z|<R} |\phi(z)|\,dx\,dy < \frac{\varepsilon}{R_0^2}, \quad \text{if } R \geq R_0.$$

Therefore,

$$\left|\iint_U \frac{\overline{\phi(z)}}{|\phi(z)|}\phi(Rz)\,dx\,dy\right| \geq \left|\iint_{|z|<R_0} \frac{\overline{\phi(z)}}{\phi(z)}\phi(Rz)\,dx\,dy\right| - \frac{\varepsilon}{R_0^2}, \qquad R \geq R_0.$$

Since $\phi(Rz)$ is bounded in $\{|z| \leq R_0\}$, uniformly with respect to R, $R_0 \leq R < 1$, we have

$$\varliminf_{R\to 1}\left|\iint_U \frac{\overline{\phi(z)}}{|\phi(z)|}\phi(Rz)\,dx\,dy\right| \geq \iint_{|z|<R_0} |\phi(z)|\,dx\,dy - \frac{\varepsilon}{R_0^2}.$$

Hence

$$\lim_{R\to 1}\left|\iint_U \frac{\overline{\phi(z)}}{|\phi(z)|}\phi(Rz)\,dx\,dy\right| = \|\phi(z)\|.$$

Also,

$$\lim_{R\to 1}\iint_U |\phi(Rz)|\,dx\,dy = \|\phi(z)\|.$$

(b) Suppose $\phi(z)$ has second-order poles, but none of higher order.

$$\frac{\iint_U \frac{\overline{\phi(z)}}{|\phi(z)|}\phi(Rz)\,dx\,dy}{\|\phi(Rz)\|} = R^2 + R^2\frac{\iint_{|z|<R} \frac{\overline{\phi(z)}}{|\phi(z)|}[\phi(Rz)-\phi(z)]\,dx\,dy}{\iint_{|z|<R}|\phi(z)|\,dx\,dy}$$
$$+ R^2\frac{\iint_{R<|z|<1} \frac{\overline{\phi(z)}}{|\phi(z)|}\phi(Rz)\,dx\,dy}{\iint_{|z|<R}|\phi(z)|\,dx\,dy}.$$

The denominator on the right side goes to infinity as $R \to 1$. For the absolute values of the terms in the numerator, we have

$$I_1(R) = \left|\iint_{|z|<R} \frac{\overline{\phi(z)}}{|\phi(z)|}[\phi(Rz)-\phi(z)]\,dx\,dy\right| \leq \iint_{|z|<R} |\phi(Rz)-\phi(z)|\,dx\,dy$$
$$\leq \int_0^R r\,dr\int_0^{2\pi} d\theta\int_{Rr}^{r} dt|\phi'(re^{i\theta})|,$$
$$I_2(R) = \left|\iint_{R<|z|<1} \frac{\overline{\phi(z)}}{|\phi(z)|}\phi(Rz)\,dx\,dy\right| \leq \frac{1}{R^2}\iint_{R^2<|z|<R} |\phi(z)|\,dx\,dy.$$

Now, if $a > 0$, $m > 2$, then

$$\text{(2.1)}\quad \int_{-a}^{a}\frac{d\theta}{|1-re^{i\theta}|^m} = \frac{p}{(1-r)^{m-1}} + o\left(\frac{1}{(1-r)^{m-1}}\right), \quad \text{as } r\to 1\ (0<r<1),$$

where p is a positive constant, not depending on a. Using (2.1), one finds that

$$\lim_{R\to 1} I_1(R) = \lim_{R\to 1} I_2(R) = 0,$$

and (1.3) therefore follows.

(c) Suppose the highest order poles of $\phi(z)$ are of order m, $m \geq 3$. We first derive a lemma.

LEMMA.[1] *Suppose* $H = \{\operatorname{Re} z > 0\}$, $g \in \mathfrak{B}(H)$, $\alpha > 0$. *Then*

$$\left| \iint_H e^{2i(1-\alpha/\pi)\theta} g(re^{i\theta})\, dr\, d\theta \right| \le \frac{|\sin \alpha|}{\alpha} \iint_H |g(z)|\, dx\, dy. \tag{2.2}$$

To prove this, let $w = (\alpha/\pi) \log z$ map H onto the strip $\Sigma = \{w = u + iv\colon -\infty < u < \infty, -\alpha/2 < v < \alpha/2\}$. In Σ, define $h(w)$ by

$$h(w)\, dw^2 = g(z)\, dz^2,$$

that is,

$$h(w) = (\pi^2/\alpha^2) e^{2\pi w/\alpha} g(e^{\pi w/\alpha}).$$

Then,

$$A = \iint_H \frac{\overline{z^{\alpha/\pi} z}}{z^{\alpha/\pi} \bar{z}} g(z)\, dx\, dy = \int_{-\alpha/2}^{\alpha/2} e^{-2iv}\, dv \int_{-\infty}^{\infty} h(u+iv)\, du,$$

while

$$B = \iint_H |g(z)|\, dx\, dy = \iint_\Sigma |h(w)|\, du\, dv.$$

Since $B < \infty$, it follows firstly, that $\int_{-\infty}^{\infty} h(u+iv)\, du$ exists for a.a.v., $-\alpha/2 < v < \alpha/2$, and secondly, that there exist sequences $\{u_n\}$, $u_n \uparrow \infty$, and $\{u'_n\}, u'_n \downarrow -\infty$, such that

$$\lim \int_{-\alpha/2}^{\alpha/2} |h(u_n + iv)|\, dv = \lim \int_{-\alpha/2}^{\alpha/2} |h(u'_n + iv)|\, dv = 0.$$

Since h is analytic in Σ, we therefore conclude by Cauchy's theorem, that

$$\int_{-\infty}^{\infty} h(u+iv)\, du = c = \text{const} \quad \text{for a.a.v.}$$

Hence,

$$A = c \int_{-\alpha/2}^{\alpha/2} e^{-2iv}\, dv = -c \sin \alpha.$$

Also,

$$|c| \le \int_{-\infty}^{\infty} |h(u+iv)|\, du \quad \text{a.a.v.}$$

Therefore $\alpha|c| \le B$, and $|A| \le (B/\alpha)|\sin \alpha|$, which proves (2.2).

Let $\alpha = ((\mu - 2)/2)\pi$. If we map $\{|\varsigma| < 1\}$ onto $\{\operatorname{Re} z > 0\}$ and transfer $g(z)\, dz^2$ as a quadratic differential, (2.2) becomes

$$\left| \iint_U \frac{|1+z|^{4-\mu} |1-z|^{\mu}}{(\overline{1+z})^{4-\mu} (\overline{1-z})^{\mu}} g(z)\, dx\, dy \right| \le \frac{|\sin((\mu-2)/2)\pi)|}{((\mu-2)/2)\pi} \iint_U |g(z)|\, dx\, dy, \tag{2.3}$$

which holds whenever $\mu > 2$, and $g \in \mathfrak{B}(U)$.

[1] This is a generalization of a result found in [**3**]. The bound is sharp for every α.

Turning now to the proof for case (c), by contradiction, suppose (1.3) held. By considering (2.1) we see that the effect of poles of order m predominates as $R \to 1$. Say, there is a pole of order m at $z = 1$. Then it is necessarily true that

$$\lim_{R\to 1} \frac{|\iint_S \frac{\overline{\phi(z)}}{|\phi(z)|}\phi(Rz)\,dx\,dy|}{\iint_S |\phi(Rz)|\,dx\,dy} = 1 \tag{2.4}$$

for any set $S = \{|\arg z| < \delta,\ \rho < |z| < 1\}$ $(\delta > 0, 0 < \rho < 1)$.

The function $\phi(z)$ has the form

$$\phi(z) = (z-1)^{-m}F(z),$$

where $F(z)$ is holomorphic and, say, nonvanishing in $\overline{S}$. If $G(z)$ is any other function holomorphic and nonvanishing in $\overline{S}$, and $G(1) = F(1)$, and if ϕ is replaced by

$$\psi(z) = (z-1)^{-m}G(z), \qquad z \in S,$$

then the quantities

$$\iint_S \frac{\overline{\phi(z)}}{|\phi(z)|}\phi(Rz)\,dx\,dy - \iint_S \frac{\overline{\psi(z)}}{|\psi(z)|}\psi(Rx)\,dx\,dy$$

and

$$\iint_S |\phi(Rz) - \psi(Rz)|\,dx\,dy$$

are negligible compared to $\iint_S |\phi(Rz)|\,dx\,dy$ as $R \to 1$. Therefore (2.4) must hold with ϕ replaced by ψ. Moreover, if $\psi(z)$ is actually holomorphic in $\overline{U}$ except for poles on ∂U, and if all poles of ψ on ∂U, except that at $z = 1$, are of strictly lower order than m, then by the same reasoning as the one leading from (1.3) to (2.4), we conclude that, necessarily,

$$\lim_{R\to 1} |\frac{|\iint_U \frac{\overline{\psi(z)}}{|\psi(z)|}\psi(Rz)\,dx\,dy|}{\iint_U |\psi(Rz)|\,dx\,dy} = 1.$$

Choosing

$$\psi(z) = \frac{(-1)^m 2^{4-m}}{(1+z)^{4-m}(1-z)^m},$$

this contradicts (2.3), however.

3. Affine mappings. Let Ω be a simply connected planar region of hyperbolic type, but not necessarily of finite area $|\Omega|$. Introducing the auxiliary conformal mapping $z = F(\varsigma)$, of $\{|\varsigma| < 1\}$ onto Ω, and using (1.2), one notes that a necessary and sufficient condition that the affine stretch

$$A_k(z) = Kx + iy, \qquad z \in \Omega,\ K > 1,\ z = x + iy$$

is an extremal mapping (among the class of quasiconformal mappings of Ω with the same prime-end boundary values as A_k) is that there exists a sequence $\phi_n \in \mathfrak{B}(\Omega)$, such that

$$\lim_{n\to\infty} \frac{\iint_\Omega \phi_n(z)\,dx\,dy}{\|\phi_n(z)\|} = 1. \tag{3.1}$$

We refer to $\{\phi_n\}$ satisfying (3.1) as a *Hamilton sequence for* Ω. If $|\Omega| < \infty$, then $\phi_n(z) \equiv 1$, $n = 1, 2, \ldots$, constitutes a Hamilton sequence, but when $|\Omega| = \infty$, the constant 1 does not belong to $\mathfrak{B}(\Omega)$, and, in fact, there may be no Hamilton sequence since A_k need not be extremal. (This occurs, for example, when Ω is a half-plane.)

In order to form a *putative* Hamilton sequence for Ω, take $0 < R < 1$, $R = R_n \uparrow 1$, and let

$$\phi_R(z) = \Phi_R'(z)^2 \quad \text{where } \Phi_R(z) = F(RF^{-1}(z)), \; z \in \Omega. \tag{3.2}$$

Since Φ_R maps Ω conformally onto a subregion of finite area, it is evident that $\phi_R(z) \in \mathfrak{B}(\Omega)$, $0 < R < 1$. Moreover, clearly,

$$\lim_{R \to 1} \phi_R(z) = 1 \quad \text{pointwise for all } z \in \Omega.$$

The obvious question is: *For which Ω is the putative sequence $\{\phi_R(z)\}$ actually a Hamilton sequence*? We can say the following:

THEOREM 3. *Suppose the mapping function $z = F(\varsigma)$ of $\{|\varsigma| < 1\}$ onto Ω is such that $F'(\varsigma)^2$ is holomorphic for $|\varsigma| \leq 1$ except for a finite number of poles on $\{|\varsigma| = 1\}$. Then the putative sequence $\{\phi_R(z)\}$, $R \uparrow 1$, is a Hamilton sequence if and only if the poles are at most of order* 2, *and this occurs if and only if $A_K(z)$ is a uniquely extremal quasiconformal mapping of Ω.*

PROOF. If we transfer $A_K(z)$ to $\{|\varsigma| < 1\}$ by means of F, the question becomes that of determining whether a Teichmüller mapping with complex dilatation

$$k\frac{\overline{F'(\varsigma)^2}}{|F'(\varsigma)|^2} \qquad \left(k = \frac{K-1}{K+1}\right)$$

is extremal or uniquely extremal, as the case may be, and whether $\{R^2F'(R\varsigma)^2\}$, $R = R_n \uparrow 1$, constitutes a corresponding Hamilton sequence. Since the factor R^2 is immaterial, the assertion is immediate by Theorem 1$'$.

EXAMPLE. Let Ω be the strip

$$\Sigma_0 = \{z = x + iy \colon\; -1 < y < 1\}.$$

Here,

$$F(\varsigma) = \frac{2}{\pi}\log\frac{1+\varsigma}{1-\varsigma}, \qquad F'(\varsigma)^2 = \frac{16}{\pi^2(1+\varsigma)^2(1-\varsigma)^2} \qquad (|\varsigma| < 1).$$

By Theorem 3, $\{\phi_R(z)\}$, as defined by (3.2), is a Hamilton sequence. The fact that the affine stretch of Σ_0 is a uniquely extremal mapping was first proved in [**5**]. For application below, we note that a Hamilton sequence $\{\sigma_n(z)\}$ for Σ_0 can also be obtained explicitly as follows. Let

$$\sigma_n(z) = (1/n)\exp(-z^2/n^2). \tag{3.3}$$

One finds that

$$\iint_{\Sigma_0} \sigma_n(z)\,dx\,dy = 2\sqrt{\pi}, \tag{3.4}$$

while

$$\iint_{\Sigma_0} |\sigma_n(z)|\, dx\, dy = \sqrt{\pi} \int_{-1}^{1} \exp\left(\frac{y^2}{n^2}\right) dy \to 2\sqrt{\pi}. \tag{3.5}$$

It is evident that together with a Hamilton sequence $\{\phi_n(z)\}$ for Σ_0, $\{\phi_n(z-\lambda_n)\}$ is also a Hamilton sequence, for any sequence of real numbers $\{\lambda_n\}$.

4. The chimney region. The chimney region

$$\mathfrak{C} = \{z\colon \operatorname{Re} z < 0\} \cup \Sigma_0$$

was introduced by Strebel who showed, with extremal-length methods [**5**], that $A_K(z)$ is extremal for $\mathfrak{C}$ but not uniquely extremal.

In order to approach the problem of constructing Hamilton sequences for $\mathfrak{C}$ we need to first consider some qualitative properties of the functions mapping $\mathfrak{C}$ conformally onto a half-plane and a disk. In fact, these mappings can be expressed in "finite form" as Schwarz-Christoffel transformations in terms of elementary functions.

Let $z(\gamma)$ be a conformal mapping of the upper half-plane $\{\operatorname{Im}\gamma > 0\}$ onto $\mathfrak{C}$. In view of the symmetry of $\mathfrak{C}$ we can preassign $z(-1) = -i$, $z(1) = i$, $z(0) =$ the point ∞ at the right end of Σ_0, $z(\infty) =$ the point ∞ in the half-plane $\{\operatorname{Re} z < 0\}$. On the basis of the angles involved we find that

$$\left(\frac{dz}{d\gamma}\right)^2 = c\frac{1-\gamma^2}{\gamma^2}, \tag{4.1}$$

where c is a positive constant. Since for every semicircle $0 \le \arg\gamma \le \pi$, $|\gamma| = h$, $(0 < h < 1)$, one has $\operatorname{Im} z(h) = 1$, $\operatorname{Im} z(-h) = -1$, it follows that $c = 4/\pi^2$. Integrating, we obtain

$$z = \frac{1}{\pi}\log\frac{\sqrt{1-\gamma^2}+1}{\sqrt{1-\gamma^2}-1} - \frac{2}{\pi}\sqrt{1-\gamma^2}, \qquad \operatorname{Im}\gamma > 0, \tag{4.2}$$

where $\sqrt{1-\gamma^2}$ is the branch with positive real part, and the logarithm term goes to zero for $\gamma = iM$, $M \to +\infty$.

THEOREM 4. (i) *Let $T(z)$ map $\mathfrak{C}$ conformally onto Σ_0, $T(i) = i$, $T(-i) = -i$, $T(+\infty) = +\infty$, and let $\{\sigma_n(z)\}$ be the Hamilton sequence* (3.3) *for Σ_0. Then*

$$q_n(z) = \sigma_n[T(z) - n^3]$$

is a Hamilton sequence for $\mathfrak{C}$.

(ii) *Let $F(\varsigma)$ map $\{|\varsigma| < 1\}$ conformally onto $\mathfrak{C}$. The sequence* (3.2), *with $R = R_n \uparrow 1$, $z \in \mathfrak{C}$, is not a Hamilton sequence for $\mathfrak{C}$.*

PROOF (i). The mapping T is given by

$$w = T(z) = \frac{2}{\pi}\log\frac{i}{\gamma(z)}.$$

Therefore,

$$\left(\frac{dz}{dw}\right)^2 = 1 + e^{-\pi w}, \qquad w \in \Sigma_0.$$

Hence,

$$\iint_{\mathbb{C}} q_n(z)\,dx\,dy = \iint_{\Sigma_0} \sigma_n(w - n^3)|1 + e^{-\pi w}|\,du\,dv, \qquad w = u + iv. \tag{4.3}$$

A straightforward estimate shows that the choice of the term n^3 guarantees, together with the rapid decay of the σ_n functions, that

$$\lim_{n\to\infty} \iint_{\substack{w\in\Sigma_0 \\ \operatorname{Re} w<A}} \sigma_n(w - n^3)|1 + e^{-\pi w}|\,du\,dv = 0,$$

and that consequently the term $|1 + e^{-\pi w}|$ behaves like the constant 1 in its effect as $n \to \infty$. Therefore, just as for (3.4), (3.5), one obtains

$$\lim_{n\to\infty} \iint_{\mathbb{C}} q_n(z)\,dx\,dy = \lim_{n\to\infty} \iint_{\mathbb{C}} |q_n(z)|\,dx\,dy = 2\sqrt{\pi}.$$

(ii) The mapping $F(\varsigma)$ is obtained by composition of $\gamma(z)$ with the Möbius transformation,

$$i\gamma = (\varsigma - 1)/(\varsigma + 1).$$

Therefore,

$$F'(\varsigma)^2 = \frac{32}{\pi^2} \frac{\varsigma^2 + 1}{(\varsigma - 1)^2(\varsigma + 1)^4}.$$

Since there is a pole of fourth order, the putative sequence is not a Hamilton sequence in this case. Attention should be called to the fact that an alternative construction of a Hamilton sequence for the affine stretch of the chimney region is possible by analyzing the proof of Ortel's Theorem 1 in [**6**].

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