WEAK* CONVERGENCE IN HIGHER DUALS
OF ORLICZ SPACES

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ABSTRACT. It is shown that the spaces \((\Sigma \Theta E)_{l^\infty(\Gamma)}\) are Grothendieck spaces for a class of Banach lattices \(E\) which includes the Orlicz spaces with weakly sequentially complete duals.

A Banach space is said to be a Grothendieck space if weak* and weak sequential convergence coincide in the dual. The simplest nontrivial example of a Grothendieck space is \(l^\infty\). In [7], the question of when the space \((E \Theta l^p)_{l^\infty(\Gamma)}\) is Grothendieck is treated. In particular, it is shown there that \((E \Theta l^p)_{l^\infty(\Gamma)}\) is Grothendieck if \(2 \leq p \leq \infty\) and \(\Gamma\) is countable. In this paper, we extend this result to a class of Banach lattices which includes the Orlicz spaces with weakly sequentially complete duals. We close these introductory remarks by mentioning that H. P. Lotz [6] has shown recently that the weak \(L^p\) spaces are Grothendieck spaces.

1. Let us start by fixing some notation. Let \(E\) be a (real) Banach lattice, \(\Gamma\) an arbitrary index set, and \(F = (E \Theta E)_{l^\infty(\Gamma)}\). For \(x \in F\), we write \(x = (x(\gamma))\), where \(x(\gamma) \in E\) for every \(\gamma \in \Gamma\). If \(x' \in F'\) and \(A \subset \Gamma\), define \(x' x_A \in F'\) by \(\langle x, x' x_A \rangle = \langle x_A, x' \rangle\) for all \(x \in F\). It is easily seen that the equation \(\mu_{x'}(A) = \|x' x_A\|\) defines a finitely additive measure on \(\Gamma\); consequently, we may identify \(\mu_{x'}\) with an element of \(l^\infty(\Gamma)'\).

**Lemma 1.** If \((x'_n)\) is a positive weak* null sequence in \(F'\), then \((\mu_{x'_n})\) is relatively weakly compact in \(l^\infty(\Gamma)'\).

**Proof.** Let \(\mu_i = \mu_{x'_i}\). If \((\mu_i)\) is not relatively weakly compact, then there exist a partition \((A_i)\) of \(\Gamma\) and \(\varepsilon > 0\) such that \(\mu_i(A_i) > \varepsilon\) for all \(i\). By definition of \(\mu_i\), there is a positive normalized sequence \((x_i)\) of \(F\) such that \(x_i x_{A_i} = 0\) and \(\langle x_i, x_i' \rangle > \varepsilon\) for all \(i\). Let \(x = \sup_i x_i\). Then \(\|x\| = 1\) and \(\langle x, x_i' \rangle > \varepsilon\) for all \(i\), contrary to the fact that \((x'_i)\) is weak* null.

**Theorem 2.** Let \(E\) be a Banach lattice with positive cone \(E_+\). Suppose there exist a function \(\tau: E_+ \to [0, \infty]\) and a positive real number \(M\) with the following properties:
1. \(\tau(0) = 0\);
2. \(\|x\| \leq 1 \Rightarrow \tau(x) \leq M\);
3. For every disjoint sequence \((x_i)_{i=1}^n \subset E_+, \sum_{i=1}^n \tau(x_i) \leq M \tau(\sum_{i=1}^n x_i);\) and
4. For every sequence \((x_i)_{i=1}^\infty \subset E_+\) with \(\sum_i \tau(x_i) \leq 1\), \(\sup_i x_i\) exists and \(\|\sup_i x_i\| \leq M\).

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797
Then, for any index set \( \Gamma \), every disjoint positive weak* null sequence \((x'_i)\) in \( F = (\Sigma \oplus E)_{\ell^\infty(\Gamma)} \) has a weakly Cauchy subsequence.

PROOF. Assume the contrary. We obtain a disjoint positive weak* null sequence \((x'_i)\) which is not weakly sequentially precompact. By Rosenthal's theorem, we may assume that \((x'_i)\) is equivalent to the \( l^1 \) basis. Since \((x'_i)\) is lattice isomorphic to \( l^1 \), there exist \( \varepsilon > 0 \) and a positive sequence \((x_{ij})_{i,j \geq 1} \subset F\) with the following properties:

(a) For every \( i \), \((x_{ij})_{1 \leq j \leq i}\) is a pairwise disjoint sequence such that \( \| \sum_{j \leq i} x_{ij} \| < 1 \); and

(b) \( \langle x_{ij}, x'_j \rangle > \varepsilon \) for \( 1 \leq j \leq i \).

Define \( A_{ij} \subseteq \Gamma \) by \( A_{ij} = \{ \gamma | \tau(x_{ij}(\gamma)) \geq 1/\sqrt{i} \} \). Note that \( \| \sum_{j \leq i} x_{ij} \| < 1 \Rightarrow \| \sum_{j \leq i} x_{ij}(\gamma) \| < 1 \) for all \( \gamma \Rightarrow \tau(\sum_{j \leq i} x_{ij}(\gamma)) \leq M \). Hence \( \sum_{j \leq i} \tau(x_{ij}(\gamma)) \leq M^2 \) since the \( x_{ij} \)'s are disjoint. Thus

\[
\bigcap_{j \in B} A_{ij} = \emptyset
\]

for all \( B \subseteq \{1, 2, \ldots, i\} \) with \( \text{card } B > M^2 \sqrt{i} \). Recall the sequence \((\mu_i)\) as defined in the proof of Lemma 1. Fix \( i \) and let \( C_i = \{ j \leq i | \mu_j(A_{ij}) < \varepsilon/2 \} \). For \( j \in C_i \), we let \( z_j = x_{ij} \chi_{A_{ij}} \), then

\[
\langle z_j, x'_j \rangle \geq \langle x_{ij}, x'_j \rangle - \langle x_{ij}, x'_j \chi_{A_{ij}} \rangle \geq \varepsilon - \| x_{ij} \| \mu_j(A_{ij}) \geq \varepsilon/2
\]

while \( \tau(z_j(\gamma)) \leq 1/\sqrt{i} \) for all \( \gamma \) by definition of \( A_{ij} \). If \((\text{card } C_i)_{i=1}^\infty\) is unbounded, there exists an infinite subset \( I \) of \( \mathbb{N} \) such that for every \( i \in I \), there exists \( j_i \in C_i \) with the \( j_i \)'s distinct for different \( i \)'s. Without loss of generality, we may also assume that \( \sum_{i \in I} 1/\sqrt{i} \leq 1 \). Choose \( z_{j_i} \) as given above. Since

\[
\sum_{i} \tau(z_{j_i}(\gamma)) \leq \sum_{i \in I} \frac{1}{\sqrt{i}} \leq 1
\]

for all \( \gamma \), \( z(\gamma) \equiv \sup_{i} z_{j_i}(\gamma) \) exists for all \( \gamma \) and \( \| z(\gamma) \| \leq M \) by property (4). Hence \( z \equiv (z(\gamma)) \in F \). However,

\[
\langle z, x'_j \rangle \geq \langle z_{j_i}, x'_j \rangle \geq \varepsilon/2
\]

for all \( i \in I \), contrary to the fact that \((x'_i)\) is weak* null. Hence \((\text{card } C_i)_{i=1}^\infty\) is bounded by some constant \( K < \infty \). Now \((\mu_i)\) is relatively weakly compact in the AL-space \( l^\infty(\Gamma)' \) by Lemma 1, hence there exists \( 0 \leq \mu \in l^\infty(\Gamma)' \) such that \( (\mu_i) \subset [0, \mu] + (\varepsilon/4)U \), where \( U \) denotes the unit ball of \( l^\infty(\Gamma)' \). Let \( D_i = \{ j \leq i | \mu_j(A_{ij}) \geq \varepsilon/2 \} \) for every \( i \). By the above, \( \text{card } D_i \geq i - K \) for all \( i \). Also \( \mu(A_{ij}) \geq \varepsilon/4 \) for all \( j \in D_i \). Using equation (*), we see that

\[
\sum_{j \in D_i} \mu(A_{ij}) \leq M^2 \sqrt{i} \mu(\Gamma)
\]

for all \( i \) and hence \( \mu(\Gamma) \geq (\varepsilon/4M^2 \sqrt{i}) \text{card } D_i \geq (\varepsilon/4M^2 \sqrt{i})(i - K) \) for all \( i \). This contradiction proves the theorem.
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THEOREM 3. Let $E$ be a countably order complete Banach lattice which satisfies a nontrivial upper estimate. If there is a function $\tau$ on $E$ as in Theorem 2, then $F = (\Sigma \oplus E)_{l^\infty(\Gamma)}$ is a Grothendieck space.

PROOF. Because of the upper estimate condition on $E$, $F'$ is weakly sequentially complete. By [2], it suffices to show that any disjoint positive weak* null sequence in $F'$ is weakly null. But this follows from Theorem 2 and the weak sequential completeness of $F'$.

REMARK. Some condition in addition to the countable order completeness and the upper estimate has to be imposed on $E$ in order for the conclusion of Theorem 3 to hold. In [3], a sequence of finite dimensional lattices $(E_n)$ which satisfy a uniform upper $p$-estimate is constructed such that $F \equiv (\Sigma \oplus E_n)_{l^\infty(\Gamma)}$ is not Grothendieck. Hence $E \equiv (\Sigma \oplus F)_{l^2}$ satisfies an upper $p$-estimate and is obviously order complete while $(\Sigma \oplus E)_{l^\infty(\Gamma)}$ is not Grothendieck.

COROLLARY 4. Under the hypotheses of Theorem 3, all the even duals of $E$ are Grothendieck spaces.

PROOF. By [1, Proposition 1.20], $E''$ is isomorphic to a complemented subspace of some ultraproduct $E_{\mathcal U}$; hence $E''$ is a quotient space of some $(\Sigma \oplus E)_{l^\infty(\Gamma)}$. Simple induction now shows that all even duals of $E$ are quotients of (different) $(\Sigma \oplus E)_{l^\infty(\Gamma)}$. But quotients of Grothendieck spaces are themselves Grothendieck.

2. We now apply the results in §1 to Orlicz spaces.

DEFINITION 5. An Orlicz function $\varphi$ is a continuous nondecreasing and convex function defined for $t > 0$ such that $\varphi(0) = 0$ and $\lim_{t \to \infty} \varphi(t) = \infty$.

DEFINITION 6. Let $(\Omega, \Sigma, \mu)$ be a measure space and let $\varphi$ be an Orlicz function, the space $L^\varphi(\Omega, \Sigma, \mu)$ is the Banach space consisting of all measurable functions $f$ such that $\int \varphi(|f(x)|/\rho) d\mu(x) < \infty$ for some $\rho > 0$ with the norm $\|f\| = \inf \left\{ \rho > 0 \left| \int \varphi(|f(x)|/\rho) d\mu(x) \leq 1 \right. \right\}$.

For details concerning Orlicz spaces, we refer the reader to [4, 5]. Here, we only wish to point out that (1) every Orlicz space is obviously order complete, and (2) if an Orlicz space $L^\varphi$ has a weakly sequentially complete dual, then it satisfies a nontrivial upper estimate. Now, if we define $\tau : (L^\varphi)_+ \to [0, \infty]$ by $\tau(f) = \int \varphi(f(x)) d\mu(x)$, then it is easily seen that $\tau$ satisfies the conditions in Theorem 2. Hence, by Theorem 3, we get

THEOREM 6. If $L^\varphi$ has a weakly sequentially complete dual, then $(\Sigma \oplus L^\varphi)_{l^\infty(\Gamma)}$ is Grothendieck for every index set $\Gamma$. Consequently, all even duals of $L^\varphi$ are Grothendieck.

REMARK. For $1 \leq p < \infty$, if we let $\varphi(t) = t^p$, then $L^\varphi = L^p$. Thus the results of Theorem 6 apply in particular to $L^p$ for $1 < p < \infty$.

REFERENCES


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