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WEAK* CONVERGENCE IN HIGHER DUALS OF ORLICZ SPACES

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ABSTRACT. It is shown that the spaces $(\Sigma\oplus E)_{l^\infty(\Gamma)}$ are Grothendieck spaces for a class of Banach lattices E which includes the Orlicz spaces with weakly sequentially complete duals.

A Banach space is said to be a Grothendieck space if weak* and weak sequential convergence coincide in the dual. The simplest nontrivial example of a Grothendieck space is l^∞. In [**7**], the question of when the space $(\Sigma \oplus E)_{l^\infty(\Gamma)}$ is Grothendieck is treated. In particular, it is shown there that $(\Sigma \oplus L^p)_{l^\infty(\Gamma)}$ is Grothendieck if $2 \leq p \leq \infty$ and Γ is countable. In this paper, we extend this result to a class of Banach lattices which includes the Orlicz spaces with weakly sequentially complete duals. We close these introductory remarks by mentioning that H. P. Lotz [**6**] has shown recently that the weak L^p spaces are Grothendieck spaces.

1. Let us start by fixing some notation. Let E be a (real) Banach lattice, Γ an arbitrary index set, and $F = (\Sigma \oplus E)_{l^\infty(\Gamma)}$. For $x \in F$, we write $x = (x(\gamma))$, where $x(\gamma) \in E$ for every $\gamma \in \Gamma$. If $x' \in F'$ and $A \subset \Gamma$, define $x'\chi_A \in F'$ by $\langle x, x'\chi_A\rangle = \langle x\chi_A, x'\rangle$ for all $x \in F$. It is easily seen that the equation $\mu_{x'}(A) = \|x'\chi_A\|$ defines a finitely additive measure on Γ; consequently, we may identify $\mu_{x'}$ with an element of $l^\infty(\Gamma)'$.

LEMMA 1. *If (x_i') is a positive weak* null sequence in F', then $(\mu_{x_i'})$ is relatively weakly compact in $l^\infty(\Gamma)'$.*

PROOF. Let $\mu_i = \mu_{x_i'}$. If (μ_i) is not relatively weakly compact, then there exist a partition (A_i) of Γ and $\varepsilon > 0$ such that $\mu_i(A_i) > \varepsilon$ for all i. By definition of μ_i, there is a positive normalized sequence $(x_i) \subset F$ such that $x_i\chi_{A_i^c} = 0$ and $\langle x_i, x_i'\rangle > \varepsilon$ for all i. Let $x = \sup_i x_i$. Then $\|x\| = 1$ and $\langle x, x_i'\rangle > \varepsilon$ for all i, contrary to the fact that (x_i') is weak* null.

THEOREM 2. *Let E be a Banach lattice with positive cone E_+. Suppose there exist a function $\tau\colon E_+ \to [0,\infty]$ and a positive real number M with the following properties:*
(1) $\tau(0) = 0$;
(2) $\|x\| \leq 1 \Rightarrow \tau(x) \leq M$;
(3) *For every disjoint sequence $(x_i)_{i=1}^n \subset E_+$, $\sum_{i=1}^n \tau(x_i) \leq M\tau(\sum_{i=1}^n x_i)$; and*
(4) *For every sequence $(x_i)_{i=1}^\infty \subset E_+$ with $\sum_i \tau(x_i) \leq 1$, $\sup_i x_i$ exists and $\|\sup_i x_i\| \leq M$.*

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Then, for any index set Γ, every disjoint positive weak null sequence (x'_i) in $F = (\Sigma \oplus E)_{l^\infty(\Gamma)}$ has a weakly Cauchy subsequence.*

PROOF. Assume the contrary. We obtain a disjoint positive weak* null sequence (x'_i) which is not weakly sequentially precompact. By Rosenthal's theorem, we may assume that (x'_i) is equivalent to the l^1 basis. Since (x'_i) is lattice isomorphic to l^1, there exist $\varepsilon > 0$ and a positive sequence $(x_{ij})_{i\geq j} \subset F$ with the following properties:

(a) For every i, $(x_{ij})_{1\leq j\leq i}$ is a pairwise disjoint sequence such that $\| \sum_{j\leq i} x_{ij} \| < 1$; and

(b) $\langle x_{ij}, x'_j \rangle > \varepsilon$ for $1 \leq j \leq i$.

Define $A_{ij} \subseteq \Gamma$ by $A_{ij} = \{\gamma | \tau(x_{ij}(\gamma)) \geq 1/\sqrt{i}\}$. Note that $\| \sum_{j\leq i} x_{ij} \| < 1 \Rightarrow \| \sum_{j\leq i} x_{ij}(\gamma) \| < 1$ for all $\gamma \Rightarrow \tau(\sum_{j\leq i} x_{ij}(\gamma)) \leq M$. Hence $\sum_{j\leq i} \tau(x_{ij}(\gamma)) \leq M^2$ since the x_{ij}'s are disjoint. Thus

$$(*) \qquad \bigcap_{j\in B} A_{ij} = \varnothing$$

for all $B \subseteq \{1, 2, \ldots, i\}$ with card $B > M^2\sqrt{i}$. Recall the sequence (μ_i) as defined in the proof of Lemma 1. Fix i and let $C_i = \{j \leq i | \mu_j(A_{ij}) < \varepsilon/2\}$. For $j \in C_i$, we let $z_j = x_{ij}\chi_{A^c_{ij}}$, then

$$\langle z_j, x'_j \rangle \geq \langle x_{ij}, x'_j \rangle - \langle x_{ij}, x'_j \chi_{A_{ij}} \rangle \geq \varepsilon - \|x_{ij}\| \mu_j(A_{ij}) \geq \varepsilon/2$$

while $\tau(z_j(\gamma)) \leq 1/\sqrt{i}$ for all γ by definition of A_{ij}. If $(\operatorname{card} C_i)_{i=1}^\infty$ is unbounded, there exists an infinite subset I of $\mathbf{N}$ such that for every $i \in I$, there exists $j_i \in C_i$ with the j_i's distinct for different i's. Without loss of generality, we may also assume that $\sum_{i\in I} 1/\sqrt{i} \leq 1$. Choose z_{j_i} as given above. Since

$$\sum_i \tau(z_{j_i}(\gamma)) \leq \sum_{i\in I} \frac{1}{\sqrt{i}} \leq 1$$

for all γ, $z(\gamma) \equiv \sup_i z_{j_i}(\gamma)$ exists for all γ and $\|z(\gamma)\| \leq M$ by property (4). Hence $z \equiv (z(\gamma)) \in F$. However,

$$\langle z, x'_{j_i} \rangle \geq \langle z_{j_i}, x'_{j_i} \rangle \geq \varepsilon/2$$

for all $i \in I$, contrary to the fact that (x'_i) is weak* null. Hence $(\operatorname{card} C_i)_{i=1}^\infty$ is bounded by some constant $K < \infty$. Now (μ_i) is relatively weakly compact in the AL-space $l^\infty(\Gamma)'$ by Lemma 1, hence there exists $0 \leq \mu \in l^\infty(\Gamma)'$ such that $(\mu_i) \subset [0, \mu] + (\varepsilon/4)U$, where U denotes the unit ball of $l^\infty(\Gamma)'$. Let $D_i = \{j \leq i | \mu_j(A_{ij}) \geq \varepsilon/2\}$ for every i. By the above, $\operatorname{card} D_i \geq i - K$ for all i. Also $\mu(A_{ij}) \geq \varepsilon/4$ for all $j \in D_i$. Using equation $(*)$, we see that

$$\sum_{j\in D_i} \mu(A_{ij}) \leq M^2\sqrt{i}\mu(\Gamma)$$

for all i and hence $\mu(\Gamma) \geq (\varepsilon/4M^2\sqrt{i})\operatorname{card} D_i \geq (\varepsilon/4M^2\sqrt{i})(i - K)$ for all i. This contradiction proves the theorem.

THEOREM 3. *Let E be a countably order complete Banach lattice which satisfies a nontrivial upper estimate. If there is a function τ on E as in Theorem 2, then $F = (\Sigma \oplus E)_{l^\infty(\Gamma)}$ is a Grothendieck space.*

PROOF. Because of the upper estimate condition on E, F' is weakly sequentially complete. By [**2**], it suffices to show that any disjoint positive weak* null sequence in F' is weakly null. But this follows from Theorem 2 and the weak sequential completeness of F'.

REMARK. Some condition in addition to the countable order completeness and the upper estimate has to be imposed on E in order for the conclusion of Theorem 3 to hold. In [**3**], a sequence of finite dimensional lattices (E_n) which satisfy a uniform upper p-estimate is constructed such that $F \equiv (\Sigma \oplus E_n)_{l^\infty}$ is not Grothendieck. Hence $E \equiv (\Sigma \oplus F)_{l^2}$ satisfies an upper p-estimate and is obviously order complete while $(\Sigma \oplus E)_{l^\infty}$ is not Grothendieck.

COROLLARY 4. *Under the hypotheses of Theorem 3, all the even duals of E are Grothendieck spaces.*

PROOF. By [**1**, Proposition 1.20], E'' is isomorphic to a complemented subspace of some ultraproduct $E_{\mathscr{U}}$; hence E'' is a quotient space of some $(\Sigma \oplus E)_{l^\infty(\Gamma)}$. Simple induction now shows that all even duals of E are quotients of (different) $(\Sigma \oplus E)_{l^\infty(\Gamma)}$. But quotients of Grothendieck spaces are themselves Grothendieck.

2. We now apply the results in §1 to Orlicz spaces.

DEFINITION 5. An *Orlicz function φ is a continuous nondecreasing and convex function defined for $t \geq 0$ such that $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$.*

DEFINITION 6. Let (Ω, Σ, μ) be a measure space and let φ be an Orlicz function, the space $L^\varphi(\Omega, \Sigma, \mu)$ is the Banach space consisting of all measurable functions f such that $\int \varphi(|f(x)|/\rho)\, d\mu(x) < \infty$ for some $\rho > 0$ with the norm

$$\|f\| = \inf\left\{\rho > 0 \middle| \int \varphi(|f(x)|/\rho)\, d\mu(x) \leq 1\right\}.$$

For details concerning Orlicz spaces, we refer the reader to [**4**, **5**]. Here, we only wish to point out that (1) every Orlicz space is obviously order complete, and (2) if an Orlicz space L^φ has a weakly sequentially complete dual, then it satisfies a nontrivial upper estimate. Now, if we define $\tau\colon (L^\varphi)_+ \to [0, \infty]$ by $\tau(f) = \int \varphi(f(x))\, d\mu(x)$, then it is easily seen that τ satisfies the conditions in Theorem 2. Hence, by Theorem 3, we get

THEOREM 6. *If L^φ has a weakly sequentially complete dual, then $(\Sigma \oplus L^\varphi)_{l^\infty(\Gamma)}$ is Grothendieck for every index set Γ. Consequently, all even duals of L^φ are Grothendieck.*

REMARK. For $1 \leq p < \infty$, if we let $\varphi(t) = t^p$, then $L^\varphi = L^p$. Thus the results of Theorem 6 apply in particular to L^p for $1 < p < \infty$.

REFERENCES

1. J. Bourgain, *New classes of $\mathscr{L}^p$-spaces*, Lecture Notes in Math., vol. 889, Springer-Verlag, 1981.

2. B. Kühn, *Schwache Konvergenz in Banachverbänden*, Arch. Math. **35** (1980), 554–558.

3. D. H. Leung, *Uniform convergence of operators and Grothendieck spaces with the Dunford-Pettis property*, Ph.D. Thesis, Univ. of Illinois, 1987.
4. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces* I, *Sequence spaces*, Springer-Verlag, 1977.
5. ———, *Classical Banach spaces* II, *Function spaces*, Springer-Verlag, 1979.
6. H. P. Lotz, *Weak* convergence in the dual of weak* L^p, preprint.
7. F. Räbiger, *Beiträge zur Strukturtheorie der Grothendieck-Räume*, Sitzungsberichte d. Heidelberger Akad. d. Wissenschaften, Math.-Naturwiss. Klasse, Jg. 1985, 4. Abhandlung, Springer, 1985.

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