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NOTE ON SUBNORMAL WEIGHTED SHIFTS

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ABSTRACT. The purpose of this note is two-fold: (1) To point out that Stampfli's characterization of subnormal shifts can be reformulated in operator form which turns out to be the Spitkovskii's characterization of subnormal operators; (2) To use this reformulation to give a new proof of the subnormality of a class of shifts.

In [5], Stampfli explicitly exhibited for a subnormal shift F_1 its minimal normal extension

$$N = \begin{bmatrix} F_1 & G_2 & & 0 \\ & F_2 & G_3 & \\ & & F_3 & \ddots \\ 0 & & & \ddots \end{bmatrix},$$

where F_n is a shift with weights $\{a_1^{(n)}, a_2^{(n)}, \ldots\}$, $G_n = \operatorname{diag}\{b_1^{(n)}, b_2^{(n)}, \ldots\}$ (G_n may have more rows than columns), and these entries satisfy:

(I) $(a_j^{(n)})^2 + (b_j^{(n)})^2 - (a_{j-1}^{(n)})^2 \geq 0$ ($b_j^{(1)} = 0$ for all j),

(II) $b_j^{(n)} = 0 \Rightarrow b_{j+1}^{(n)} = 0$,

(III) there exists a constant M such that $|a_j^{(n)}| \leq M$ and $|b_j^{(n)}| \leq M$ for $n = 2, 3, \ldots$, and $j = 1, 2, \ldots$, where

$$b_j^{(n+1)} = [(a_j^{(n)})^2 + (b_j^{(n)})^2 - (a_{j-1}^{(n)})^2]^{1/2}$$

and $a_j^{(n+1)} = a_j^{(n)} b_{j+1}^{(n+1)} / b_j^{(n+1)}$ (if $b_{j_0}^{(n)} = 0$ then $a_{j_0}^{(n)}$ is taken to be zero).

Conditions (I), (II), and (III) were then shown to be sufficient as well. Interestingly, they can be reformulated in operator forms as follows:

(I′) $D_n \geq 0$,

(II′) $F_n \ker D_n \subset \ker D_n$,

(III′) $\|F_n\|$, $\|D_n\| \leq M$, where $G_{n+1} = D_n^{1/2}$, $D_1 = F_1^* F_1 - F_1 F_1^*$, $D_{n+1} = D_n|_{H_{n+1}} + F_{n+1}^* F_{n+1} - F_{n+1} F_{n+1}^*, H_{n+1} = (\operatorname{Ran} D_n)^-$, and F_{n+1} denotes the continuous extension of $D_n^{1/2} F_n D_n^{-1/2}$ to $\operatorname{Ran}(D_n^{1/2})^-$ ($= H_{n+1}$) from $\operatorname{Ran}(D_n^{1/2})$. Indeed, (I′) and (III′) are direct translation of (I) and (III), respectively. As for the equivalence of (II) and (II′), (II) implies (II′) is obvious; to see the converse, it is enough to show that no shift F_n (defined on H_n) has zero weights. But this can

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be done by an induction argument. These three conditions happen to be precisely Spitkovskii's subnormality criterion [**4**].

Next we use this reformulation to show the subnormality of a class of shifts. In [**1**], C. Cowen and J. Long constructed a subnormal Toeplitz operator that is neither analytic nor normal and thus answered negatively a question raised by Halmos [**2**]. A key step in the construction is to show:

LEMMA [**1**]. *If $0 < \alpha < 1$, the weighted shift with weights $w_n = (1-\alpha^{2n+2})^{1/2}$ for $n = 0, 1, 2, \ldots$ is subnormal.*

The proof of the lemma is based on the Berger-Gellar-Wallen criterion but involves some tricky special function theory; see reference in [**1**]. So it would be nice to have a purely operator-theoretic proof. In fact, it was Ma Ji Pu and Zhou Shao Jie [**3**] who first used the Spitkovskii's characteristion to prove the subnormality of the weighted shift with weights $w_n = (2-1/2^n)^{1/2}$ for $n = 0, 1, 2, \ldots$. Since this shift is a constant multiple of the shift ($\alpha = 1/\sqrt{2}$) in the lemma, we thought it is probably not an isolated case. It is not, as the following computation shows: Let F_1 be a shift in the lemma. Then: (I)$'$ $D_1 = F_1^*F_1 - F_1F_1^* = \operatorname{diag}\{1-\alpha^2, \alpha^2-\alpha^4, \ldots\} \geq 0$ and $D_{n+1} = D_n|_{H_{n+1}} + F_{n+1}^*F_{n+1} - F_{n+1}F_{n+1}^* = (1+\alpha^2+\cdots+\alpha^{2n})D_1 \geq 0$,

(II)$'$ $F_n \ker D_n \subset \ker D_n$ since $\ker D_n = \{0\}$,

(III)$'$ $F_{n+1} = D_n^{1/2}F_nD_n^{-1/2} = \alpha^nF_1$ and D_n are clearly bounded. Thus all three conditions are satisfied and so F_1 is subnormal.

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