SUPER-RIGID FAMILIES OF STRONGLY BLACKWELL SPACES

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Abstract. We construct a complete subfield $\mathcal{F}$ of $\mathcal{P}(\mathbb{R})$, isomorphic to $\mathcal{P}(\mathbb{R})$, of pairwise non-Borel-isomorphic rigid strong Blackwell subsets of $\mathbb{R}$ such that there are only 'very few' measurable functions between any two members of $\mathcal{F}$. As a consequence, we obtain large chains and antichains of non-isomorphic rigid strong Blackwell subsets of $\mathbb{R}$. Also, there is a collection of continuously many dense subsets of $\mathbb{R}$ such that any two of them differ only by two elements, but none of them is a continuous image of any other.

1. Introduction. In this paper, we will construct a large subfield $\mathcal{F}$ of $\mathcal{P}(\mathbb{R})$ with the following properties. Any two members $A, B \in \mathcal{F}$ which are nonempty proper subsets of $\mathbb{R}$ will be nonanalytic strong Blackwell sets (definitions below) with only 'very few' measurable functions both from $A$ into $B$ (or vice versa) or from $A$ into itself. In particular, the spaces $(A, \mathcal{B}_A)$ and $(B, \mathcal{B}_B)$ (both sets equipped with the natural Borel structure) will be nonisomorphic if $A \neq B$, and $(A, \mathcal{B}_A)$ will be a rigid Borel space in the sense of [1], i.e., any isomorphism of $(A, \mathcal{B}_A)$ onto itself moves at most countably many points. In fact, we show that for any topologically sufficiently large Blackwell subset $X$ of $\mathbb{R}$, there exists a complete subfield $\mathcal{F}$ of $\mathcal{P}(X)$ with the above properties.

Let $\mathcal{A}$ be a separable $\sigma$-algebra on a set $A$, i.e. $\mathcal{A}$ is countably generated and contains all singletons. Then $(A, \mathcal{A})$ is a Blackwell space, if the only separable substructure of $\mathcal{A}$ is $\mathcal{A}$ itself. Also, $(A, \mathcal{A})$ is a strong Blackwell space if any two countably generated sub-$\sigma$-algebras of $\mathcal{A}$ with the same atoms coincide. Subsets of $\mathbb{R}$ will always be endowed with the natural Borel-$\sigma$-field, and we say that a subset $A$ of $\mathbb{R}$ is a (strong) Blackwell set if $(A, \mathcal{B}_A)$ is a (strong) Blackwell space. Ramachandran [11] discusses strong Blackwell spaces as a natural model for probability theory on which various concepts of independence of random variables coincide. Already Blackwell [3] and also Mackey [9], in a study of group representations, showed that any analytic subset of the reals is a (strong) Blackwell set. Answering a question of Blackwell [3], Orkin [10] constructed a nonanalytic Blackwell subset of $\mathbb{R}$. Ryll-Nardzewski [13] (see also [1]) found a subset $X$ of $\mathbb{R}$ such that both $X$ and $Y = \mathbb{R} \setminus X$ are nonanalytic strong Blackwell sets; by a recent result of Shortt [16], $(X, \mathcal{B}_X)$ and $(Y, \mathcal{B}_Y)$ must then be nonisomorphic. On the other hand, Bhaskara Rao and Rao [1] constructed a subset $X$ of $\mathbb{R}$ such that $(X, \mathcal{B}_X)$ is a rigid Borel space. In the present paper, we wish to sharpen these results as indicated qualitatively above. Shortt [14] calls a subset $X$ of $\mathbb{R}$ Borel-dense, if $X$ intersects each
uncountable Borel set of \( \mathbb{R} \), and shows that under the assumption of Borel-density various Blackwell concepts coincide; in particular any Borel-dense Blackwell subset \( X \) of \( \mathbb{R} \) is strongly Blackwell. In fact, many of the strongly Blackwell spaces constructed in the literature can be shown to be Borel-dense in some suitable Polish space and hence without loss of generality in \( \mathbb{R} \), cf. \[1, 2, 5-7, 10, 13-15\].

Before stating our main result, let us introduce some more notation. Let \( X \subseteq \mathbb{R} \) and \( f : X \rightarrow \mathbb{R} \) be a measurable function. Then \( \text{supp}(f) = \{ x \in X : x \neq f(x) \} \), the support of \( f \). We call \( f \) inessential, if \( f(\text{supp}(f)) \) is countable, and otherwise \( f \) is essential. (For the origin of this notion (in a different context) we refer the reader to Dugas and Göbel \[4, p. 458\].) As there are always many inessential measurable functions acting on \( X \), we can only hope to construct subsets \( X \) of \( \mathbb{R} \) with no essential measurable mappings. Observe in the following that if \( f \) has uncountable support and, for instance, either \( f \) is injective or \( f \) maps \( X \) onto \( X \), then \( f \) is essential. A family \( \mathcal{F} \) of subsets of \( \mathbb{R} \) will be called super-rigid if for any two members \( A, B \in \mathcal{F} \) there is no essential measurable mapping \( f \) of \( A \) into \( B \). Also, we say that a subset \( A \) of \( \mathbb{R} \) is super-rigid if \( \{ A \} \) is super-rigid; then in particular \( (A, \mathcal{B}_A) \) is rigid. A subfield \( \mathcal{F} \) of \( \mathcal{P}(X) \) is complete, if it is closed under arbitrary unions. We will show:

**Theorem 1.** Let \( X \) be any Borel-dense Blackwell subset of \( \mathbb{R} \). There exists a complete subfield \( \mathcal{F} \) of \( \mathcal{P}(X) \) with the following properties:

1. Each non-empty set \( A \in \mathcal{F} \) is a Borel-dense strong Blackwell set.
2. \( \mathcal{F} \setminus \{ \emptyset, X \} \) is super-rigid.
3. \( \mathcal{F} \) is isomorphic (as a Boolean algebra) to \( \mathcal{P}(\mathbb{R}) \).

Note that in particular \( X = \mathbb{R} \) satisfies the hypothesis of Theorem 1. We list a few immediate consequences of Theorem 1. Let \( A, B \in \mathcal{F} \) be nonempty proper subsets of \( X \).

1. Both \( A \) and \( X \setminus A \) are nonanalytic (in fact, universally nonmeasurable). If \( A \not\subseteq B \), there is no measurable injection \( f : A \rightarrow B \) and also no measurable mapping of \( B \) onto \( A \), as \( A \setminus B \in \mathcal{F} \) is uncountable. Whenever \( f : A \rightarrow B \) is an order-preserving injection, then \( A \subseteq B \) and \( f \) is the identity. Hence Theorem 1 contains a classical result of Sierpiński \[17\] (cf. also \[12, §9.2\]) on rigid dense order types contained in \( (\mathbb{R}, \leq) \). Moreover, if \( f : A \rightarrow A \) is onto and nondecreasing, then \( f \) again is the identity, showing that \( (A, \leq) \) is a Hopfian order in the sense of Ash \[12, p. 155\].

2. Let \( P, Q \) be any two nonatomic measures on \( (A, \mathcal{B}_A) \) such that \( Q = f(P) \) for some measurable mapping \( f \) of \( A \) into itself. As \( f \) is inessential, it follows that \( P = Q \).

3. If \( A \) and \( B \) are disjoint, there are no two nonatomic measures \( P \) and \( Q \) on \( (A, \mathcal{B}_A) \) and \( (B, \mathcal{B}_B) \), respectively, such that \( Q \) is the image of \( P \) under some measurable mapping of \( A \) into \( B \).

4. Let \( c \) denote the cardinality of the continuum. \( \{ \mathcal{B}_A : A \in \mathcal{F} \setminus \{ \emptyset, X \} \} \) constitutes a family of \( 2^c \) pairwise nonisomorphic countably generated \( \sigma \)-algebras. Clearly, \( 2^c \) is the maximal possible size of such a family.

Further consequences of Theorem 1 are given in §2.

2. Proof of Theorem 1 and further consequences. In this section we wish to prove Theorem 1 and derive a few further consequences from it. The following
notions developed by R. M. Shortt (see, e.g., [2, 14]) will be useful for us. A subset $X$ of $\mathbb{R}$ is called Borel-dense (of order 1) in $\mathbb{R}$ if $X$ intersects each uncountable member of $\mathcal{B}_\mathbb{R}$. Any such set $X$ has cardinality $c$. A subset $R$ of $\mathbb{R} \times \mathbb{R}$ is reticulate if $R \subseteq (C \times \mathbb{R}) \cup (\mathbb{R} \times C)$ for some countable subset $C \subseteq \mathbb{R}$. A subset $X$ of $\mathbb{R}$ is Borel-dense of order 2 if $X \times X$ intersects every set $R$ in $\mathcal{B}_{\mathbb{R} \times \mathbb{R}}$ which is not reticulate; in this case $X$ is Borel-dense of order 1. We also have, among others, the following equivalences:

**Lemma 2.1 (Shortt [14]).** For any subset $X$ of $\mathbb{R}$ the following are equivalent:

1. $X$ is Borel-dense of order 2 in $\mathbb{R}$.
2. $X$ is Borel-dense and a Blackwell set.
3. $X$ is Borel-dense and a strong Blackwell set.

As usual we identify each cardinal with the least ordinal of its cardinality. We first need a few preparations.

**Lemma 2.2.** Let $X$ be a Borel-dense subset of $\mathbb{R}$ and $f: \mathbb{R} \to \mathbb{R}$ an essential measurable function. There exists a subset $A$ of $X \cap \text{supp}(f)$ of cardinality $c$ such that $f$ acts injectively on $A$.

**Proof.** Let $S = \text{supp}(f) = \{x \in \mathbb{R}: x \neq f(x)\}$, $f'$ be the restriction of $f$ to $S$, and $T = f(S) = f'(S)$. Thus $T$ is uncountable and analytic. Let $T^*$ be the set of all $t \in T$ for which $f^{-1}(\{t\})$ is uncountable. Then $T^*$ is analytic (see [8, p. 496]). If $T^*$ is uncountable, choose for each $t \in T^*$ an element $x_t \in X \cap f^{-1}(\{t\})$ and put $A = \{x_t: t \in T^*\}$. Then $|A| = |T^*| = c$. On the other hand, if $T^*$ is countable, $T \setminus T^*$ is uncountable and analytic and hence $|X \cap f^{-1}(T \setminus T^*)| = c$. As $f^{-1}(\{t\})$ is countable for each $t \in T \setminus T^*$, we can select a subset $A$ of $X \cap f^{-1}(T \setminus T^*)$ of cardinality $c$ on which $f$ acts injectively. In either case, the result follows.

**Lemma 2.3.** Let $X$ be Borel-dense of order 2 in $\mathbb{R}$ and $R$ any nonreticulate member of $\mathcal{B}_{\mathbb{R} \times \mathbb{R}}$. Then $|(X \times X) \cap R| = c$.

**Proof.** It suffices to show that there is a system $\{R_i: i < c\}$ of $c$ pairwise disjoint subsets $R_i$ of $R$, each being a nonreticulate member of $\mathcal{B}_{\mathbb{R} \times \mathbb{R}}$. Let $\pi_1$, $\pi_2$ be the canonical projection of $R$ onto its first (second) coordinate, respectively. Let $A_i = \{x \in \mathbb{R}: \pi_i^{-1}(\{x\})\}$ be uncountable; then $A_i$ is analytic ($i = 1, 2$). If $A_1$ is uncountable, we can choose a system $\{B_i: i < c\}$ of $c$ pairwise disjoint subsets $B_i$ of cardinality $c$ of $A_1$ with $B_i \in \mathcal{B}_\mathbb{R}$. Put $R_i = \pi_1^{-1}(B_i)$ for each $i < c$ to obtain the result. Therefore let us assume that $A_1$ and $A_2$ are both countable. Put $R^* = R \setminus ((A_1 \times \mathbb{R}) \cup (\mathbb{R} \times A_2))$. As $\pi_1(R^*)$ is uncountable and analytic, we can now choose $c$ pairwise disjoint subsets $B_i$ ($i < c$) of cardinality $c$ of $\pi_1(R^*)$ each belonging to $\mathcal{B}_\mathbb{R}$. Again put $R_i = \pi_1^{-1}(B_i)$ ($i < c$) to obtain the result.

If $q \in A \times B \times C \times D$ is a quadruple and, for instance, $b \in B$, we write $q = (*,b,*,*)$ to denote that $q = (a,b,c,d)$ for some $a \in A$, $c \in C$, $d \in D$. Now we show:

**Theorem 2.4.** Let $X$ be any Borel-dense Blackwell subset of $\mathbb{R}$. There exists a decomposition $X = \bigcup_{i < c} X_i$ of $X$ into $c$ pairwise disjoint subsets $X_i$ with the following properties:

1. Each subset $X_i$ ($i < c$) is Borel-dense of order 2 in $\mathbb{R}$.
(2) Whenever \( i, j < c \), \( X_i \subseteq Y \subseteq X \), and \( f : Y \rightarrow \mathbb{R} \) is an essential measurable function, then \( |f(X_i) \cap (X_j \cup (R \setminus X))| = c \).

**Proof.** Let \( \{q_\alpha : \alpha < c\} \) be a list of all quadruples \( q = (g, i, j, R) \), where \( g : \mathbb{R} \rightarrow \mathbb{R} \) is an essential measurable function, the ordinals \( i, j \) are less than \( c \), and \( R \) is a nonreticulate member of \( \mathcal{B}_{\mathbb{R} \times \mathbb{R}} \). We also write \( q_\alpha = (g_\alpha, i_\alpha, j_\alpha, R_\alpha) \). By Lemma 2.2, choose for each \( \alpha < c \) a subset \( A_\alpha \) of \( X \cap \text{supp}(g_\alpha) \) of cardinality \( c \) on which \( g_\alpha \) acts injectively. We now choose elements \( x_\alpha, u_\alpha, v_\alpha \in \mathbb{R} (\alpha < c) \) inductively as follows. Assume that \( \alpha < c \) and for each \( \beta < \alpha \) we have found elements \( x_\beta \in A_\beta \) and \( u_\beta, v_\beta \in X \) such that \( (u_\beta, v_\beta) \in R_\beta \) and the elements

\[
x_\beta, g_\beta(x_\beta), u_\beta, v_\beta \quad (\beta < \alpha)
\]
are all different from each other except that possibly \( u_\beta = v_\beta \) for some \( \beta < \alpha \).

As \( |A_\alpha| = c \), there exists \( x_\alpha \in A_\alpha \) such that \( x_\alpha \) and \( g_\alpha(x_\alpha) \) are different from all elements listed in \((*)\). Next, as \( |R_\alpha \cap (X \times X)| = c \) by Lemmas 2.1 and 2.3, we can choose \( u_\alpha, v_\alpha \in X \) different from \( x_\alpha, g_\alpha(x_\alpha) \), and all elements of \((*)\) such that \( (u_\alpha, v_\alpha) \in R_\alpha \).

Now put

\[
X'_i = \{x_\alpha, u_\alpha, v_\alpha : q_\alpha = (*, i, *, *), \alpha < c\}
\]

for each \( i < c \). By construction, these sets \( X'_i \) are pairwise disjoint. Let \( Z = X \setminus \bigcup_{i < c} X'_i \) and put \( X_0 = X'_0 \cup Z \), \( X_i = X'_i \) for each \( 0 < i < c \), where \( 0 = \min\{i : i < c\} \).

Then clearly condition (1) is satisfied.

Now let \( f : Y \rightarrow \mathbb{R} \) be any essential measurable mapping where \( X_i \subseteq Y \subseteq X \), and let \( i, j < c \). We can extend \( f \) to a measurable function \( g \) defined on all of \( \mathbb{R} \) (cf. [8, p. 434]). Then \( g \) is essential, and for any nonreticulate set \( R \) in \( \mathcal{B}_{\mathbb{R} \times \mathbb{R}} \) there exists \( \alpha = \alpha(R) < c \) with \( (g, i, j, R) = q_\alpha \). Thus \( x_\alpha \in X_i \) and \( g(x_\alpha) \in X_j \cup (R \setminus X) \). Since there are precisely \( c \) such nonreticulates \( R \), we obtain \( |f(X_i) \cap (X_j \cup (R \setminus X))| = c \).

Note that each superset (in \( \mathbb{R} \)) of any of the sets \( X_i \) of Theorem 2.4 is a strong Blackwell set by Lemma 2.1. Now we give the

**Proof of Theorem 1.** Choose a decomposition \( X = \bigcup_{i \in \mathcal{R}} X_i \) of \( X \) into \( c \) pairwise disjoint subsets \( X_i \) with all the properties of Theorem 2.4. Let \( \mathcal{F} \) be the system of all unions \( U_A = \bigcup_{i \in A} X_i \) where \( A \subseteq \mathcal{R} \). Now suppose that \( A, B \) are nonempty proper subsets of \( \mathcal{R} \) and \( f : U_A \rightarrow \mathbb{R} \) is an essential measurable function. Then \( f(U_A) \) intersects \( U_{\mathcal{R} \setminus B} \cap (X \setminus X) \). Observing Lemma 2.1, it follows that \( \mathcal{F} \) satisfies all the requirements of the theorem.

The following result, which is immediate both by Theorem 2.4 (and Lemma 2.1) and Theorem 1, generalizes Bhaskara Rao and Rao [1, Proposition 12] and Rosen-stein [12, Theorem 9.6].

**Corollary 2.5.** Let \( X \) be any Borel-dense Blackwell subset of \( \mathbb{R} \) and \( \kappa \) any finite or infinite cardinal number with \( 2 \leq \kappa \leq c \). Then there exists a decomposition \( X = \bigcup_{i < \kappa} X_i \) of \( X \) into \( \kappa \) pairwise disjoint nonanalytic super-rigid strong Blackwell subsets \( X_i \) such that, for any \( i, j < \kappa \) with \( i \neq j \), there is no measurable mapping of \( X_i \) into \( X_j \) with uncountable range.

We also note the following related result.
COROLLARY 2.6. Let \( X \) be any Borel-dense Blackwell subset of \( \mathbb{R} \). There exists a super-rigid family \( \mathcal{A} \), closed under complementation in \( X \), of \( 2^c \) (nonanalytic) strong Blackwell subsets of \( X \) such that whenever \( A, B \in \mathcal{A} \) with \( A \neq B \), there exists no measurable mapping \( f : A \rightarrow B \) which is either injective or surjective.

PROOF. First choose a subfield \( \mathcal{F} \) of \( \mathcal{P}(X) \) with the properties (1)-(3) of Theorem 1, and let \( \varphi \) be an isomorphism from \( \mathcal{P}(\mathbb{R}) \) onto \( \mathcal{F} \). Next split \( \mathbb{R} = R_1 \cup R_2 \) into two disjoint subsets of equal cardinality, and let \( f : R_1 \rightarrow R_2 \) be a bijection. For each subset \( A \subseteq R_1 \) put \( S_A = A \cup (R_2 \setminus f(A)) \). Then \( \mathcal{F} = \{S_A : A \subseteq R_1 \} \) is closed under complementation, has size \( 2^c \), and consists only of pairwise incomparable elements. Now put \( \mathcal{A} = \varphi(\mathcal{F}) \) to obtain the result.

In view of Corollary 2.5, we wish to show that \( X \) is also the union of a well ordered chain of length \( c \) of pairwise nonisomorphic strong Blackwell subsets. In fact we have:

COROLLARY 2.7. Let \( X \) be any Borel-dense Blackwell subset of \( \mathbb{R} \), and let \( (I, \leq) \) be any linearly ordered set with \( |I| \leq c \) and without a greatest or a smallest element. Then there exists a super-rigid family \( \mathcal{C} \) of strong Blackwell subsets of \( \mathbb{R} \) such that:

1. \( (\mathcal{C}, \subseteq) \) is a chain isomorphic to \( (I, \leq) \).
2. Whenever \( A, B \in \mathcal{C} \) with \( A \not\subseteq B \), there is no measurable injection of \( B \) into \( A \), and also no measurable surjection of \( A \) onto \( B \).
3. \( X = \bigcup\{A : A \in \mathcal{C}\} \) and \( \bigcap\{A : A \in \mathcal{C}\} = \emptyset \).

PROOF. Split \( \mathbb{R} = \bigcup_{i \in I} X_i \) into \( |I| \) pairwise disjoint nonempty subsets \( X_i \). For each \( i \in I \) let \( A_i = \bigcup\{X_j : j \in I, j \leq i\} \), and put \( \mathcal{F} = \{A_i : i \in I\} \). Clearly \( (I, \leq) \) and \( (\mathcal{F}, \subseteq) \) are order-isomorphic. Now let \( \mathcal{C} \) be the image of \( \mathcal{F} \) under any isomorphism from \( \mathcal{P}(\mathbb{R}) \) onto \( \mathcal{F} \), where \( \mathcal{F} \) is the subfield of \( \mathcal{P}(X) \) of Theorem 1. The result follows.

Let us say that two sets of reals have incomparable continuity type, if none of them is a continuous image of the other (cf. [8, p. 428]). The following observation is in the spirit of a result of Sierpiński [17] (cf. [12, Theorem 9.10(3)]) dealing with order types of subsets of \( \mathbb{R} \).

COROLLARY 2.8. There are \( c \) Borel-dense subsets of \( \mathbb{R} \) any two of which have incomparable continuity types but differ by only two elements.

PROOF. Let \( X, Y \) be two of the subsets \( X_i \) of \( \mathbb{R} \) of Theorem 2.4. We claim that the sets \( X_y = X \cup \{y\} (y \in Y) \) satisfy our requirements. Suppose that \( y, z \in Y \) with \( y \neq z \) and \( f : X_y \rightarrow X_z \) is continuous and onto. Let \( x \in X_y \) with \( f(x) = z \). There is a neighborhood \( U \) of \( x \) in \( X_y \) which is disjoint with \( f(U) \). Hence \( U \setminus \{y\} \subseteq f(\text{supp}(f)) \). As \( X \) is Borel-dense, \( U \) is uncountable and \( f \) is essential. But then \( |X_z \cap Y| = c \) by Theorem 2.4, a contradiction.

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