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SUPER-RIGID FAMILIES OF STRONGLY BLACKWELL SPACES

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ABSTRACT. We construct a complete subfield $\mathscr{F}$ of $\mathscr{P}(\mathbf{R})$, isomorphic to $\mathscr{P}(\mathbf{R})$, of pairwise non-Borel-isomorphic rigid strong Blackwell subsets of $\mathbf{R}$ such that there are only 'very few' measurable functions between any two members of $\mathscr{F}$. As a consequence, we obtain large chains and antichains of non-isomorphic rigid strong Blackwell subsets of $\mathbf{R}$. Also, there is a collection of continuously many dense subsets of $\mathbf{R}$ such that any two of them differ only by two elements, but none of them is a continuous image of any other.

1. Introduction. In this paper, we will construct a large subfield $\mathscr{F}$ of $\mathscr{P}(\mathbf{R})$ with the following properties. Any two members $A, B \in \mathscr{F}$ which are nonempty proper subsets of $\mathbf{R}$ will be nonanalytic strong Blackwell sets (definitions below) with only 'very few' measurable functions both from A into B (or vice versa) or from A into itself. In particular, the spaces $(A, \mathscr{B}_A)$ and $(B, \mathscr{B}_B)$ (both sets equipped with the natural Borel structure) will be nonisomorphic if $A \neq B$, and $(A, \mathscr{B}_A)$ will be a rigid Borel space in the sense of [**1**], i.e., any isomorphism of $(A, \mathscr{B}_A)$ onto itself moves at most countably many points. In fact, we show that for any topologically sufficiently large Blackwell subset X of $\mathbf{R}$, there exists a complete subfield $\mathscr{F}$ of $\mathscr{P}(X)$ with the above properties.

Let $\mathscr{A}$ be a separable σ-algebra on a set A, i.e. $\mathscr{A}$ is countably generated and contains all singletons. Then $(A, \mathscr{A})$ is a *Blackwell space*, if the only separable substructure of $\mathscr{A}$ is $\mathscr{A}$ itself. Also, $(A, \mathscr{A})$ is a *strong Blackwell space* if any two countably generated sub-σ-algebras of $\mathscr{A}$ with the same atoms coincide. Subsets of $\mathbf{R}$ will always be endowed with the natural Borel-σ-field, and we say that a subset A of $\mathbf{R}$ is a *(strong) Blackwell set* if $(A, \mathscr{B}_A)$ is a (strong) Blackwell space. Ramachandran [**11**] discusses strong Blackwell spaces as a natural model for probability theory on which various concepts of independence of random variables coincide. Already Blackwell [**3**] and also Mackey [**9**], in a study of group representations, showed that any analytic subset of the reals is a (strong) Blackwell set. Answering a question of Blackwell [**3**], Orkin [**10**] constructed a nonanalytic Blackwell subset of $\mathbf{R}$. Ryll-Nardzewski [**13**] (see also [**1**]) found a subset X of $\mathbf{R}$ such that both X and $Y = \mathbf{R} \setminus X$ are nonanalytic strong Blackwell sets; by a recent result of Shortt [**16**], $(X, \mathscr{B}_X)$ and $(Y, \mathscr{B}_Y)$ must then be nonisomorphic. On the other hand, Bhaskara Rao and Rao [**1**] constructed a subset X of $\mathbf{R}$ such that $(X, \mathscr{B}_X)$ is a rigid Borel space. In the present paper, we wish to sharpen these results as indicated qualitatively above. Shortt [**14**] calls a subset X of $\mathbf{R}$ *Borel-dense*, if X intersects each

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uncountable Borel set of $\mathbf{R}$, and shows that under the assumption of Borel-density various Blackwell concepts coincide; in particular any Borel-dense Blackwell subset X of $\mathbf{R}$ is strongly Blackwell. In fact, many of the strongly Blackwell spaces constructed in the literature can be shown to be Borel-dense in some suitable Polish space and hence without loss of generality in $\mathbf{R}$, cf. [**1**, **2**, **5**–**7**, **10**, **13**–**15**].

Before stating our main result, let us introduce some more notation. Let $X \subseteq \mathbf{R}$ and $f\colon X \to \mathbf{R}$ be a measurable function. Then $\operatorname{supp}(f) = \{x \in X\colon x \neq f(x)\}$, the *support of* f. We call f *inessential*, if $f(\operatorname{supp}(f))$ is countable, and otherwise f is *essential.* (For the origin of this notion (in a different context) we refer the reader to Dugas and Göbel [**4**, p. 458].) As there are always many inessential measurable functions acting on X, we can only hope to construct subsets X of $\mathbf{R}$ with no essential measurable mappings. Observe in the following that if f has uncountable support and, for instance, either f is injective or f maps X onto X, then f is essential. A family $\mathscr{F}$ of subsets of $\mathbf{R}$ will be called *super-rigid* if for any two members $A, B \in \mathscr{F}$ there is no essential measurable mapping f of A into B. Also, we say that a subset A of $\mathbf{R}$ is super-rigid if $\{A\}$ is super-rigid; then in particular $(A, \mathscr{B}_A)$ is rigid. A subfield $\mathscr{F}$ of $\mathscr{P}(X)$ is *complete*, if it is closed under arbitrary unions. We will show:

THEOREM 1. *Let X be any Borel-dense Blackwell subset of* $\mathbf{R}$. *There exists a complete subfield $\mathscr{F}$ of $\mathscr{P}(X)$ with the following properties:*

(1) *Each non-empty set $A \in \mathscr{F}$ is a Borel-dense strong Blackwell set.*

(2) *$\mathscr{F} \setminus \{\varnothing, X\}$ is super-rigid.*

(3) *$\mathscr{F}$ is isomorphic (as a Boolean algebra) to $\mathscr{P}(\mathbf{R})$.*

Note that in particular $X = \mathbf{R}$ satisfies the hypothesis of Theorem 1. We list a few immediate consequences of Theorem 1. Let $A, B \in \mathscr{F}$ be nonempty proper subsets of X.

(1) Both A and $X \setminus A$ are nonanalytic (in fact, universally nonmeasurable). If $A \nsubseteq B$, there is no measurable injection $f\colon A \to B$ and also no measurable mapping of B onto A, as $A \setminus B \in \mathscr{F}$ is uncountable. Whenever $f\colon A \to B$ is an order-preserving injection, then $A \subseteq B$ and f is the identity. Hence Theorem 1 contains a classical result of Sierpiński [**17**] (cf. also [**12**, §9.2]) on rigid dense order types contained in $(\mathbf{R}, \leq)$. Moreover, if $f\colon A \to A$ is onto and nondecreasing, then f is again the identity, showing that $(A, \leq)$ is a Hopfian order in the sense of Ash [**12**, p. 155].

(2) Let P, Q be any two nonatomic measures on $(A, \mathscr{B}_A)$ such that $Q = f(P)$ for some measurable mapping f of A into itself. As f is inessential, it follows that $P = Q$.

(3) If A and B are disjoint, there are no two nonatomic measures P and Q on $(A, \mathscr{B}_A)$ and $(B, \mathscr{B}_B)$, respectively, such that Q is the image of P under some measurable mapping of A into B.

(4) Let c denote the cardinality of the continuum. $\{\mathscr{B}_A\colon A \in \mathscr{F} \setminus \{\varnothing, X\}\}$ constitutes a family of 2^c pairwise nonisomorphic countably generated σ-algebras. Clearly, 2^c is the maximal possible size of such a family.

Further consequences of Theorem 1 are given in §2.

2. Proof of Theorem 1 and further consequences. In this section we wish to prove Theorem 1 and derive a few further consequences from it. The following notions developed by R. M. Shortt (see, e.g., [**2**, **14**]) will be useful for us. A subset X of $\mathbf{R}$ is called *Borel-dense* (*of order* 1) in $\mathbf{R}$ if X intersects each uncountable member of $\mathscr{B}_{\mathbf{R}}$. Any such set X has cardinality c. A subset R of $\mathbf{R}\times\mathbf{R}$ is *reticulate* if $R \subseteq (C\times\mathbf{R})\cup(\mathbf{R}\times C)$ for some countable subset $C\subseteq\mathbf{R}$. A subset X or $\mathbf{R}$ is *Borel-dense of order* 2 if $X\times X$ intersects every set R in $\mathscr{B}_{\mathbf{R}\times\mathbf{R}}$ which is not reticulate; in this case X is Borel-dense of order 1. We also have, among others, the following equivalences:

LEMMA 2.1 (SHORTT [**14**]). *For any subset X of $\mathbf{R}$ the following are equivalent:*

(1) *X is Borel-dense of order 2 in $\mathbf{R}$.*
(2) *X is Borel-dense and a Blackwell set.*
(3) *X is Borel-dense and a strong Blackwell set.*

As usual we identify each cardinal with the least ordinal of its cardinality. We first need a few preparations.

LEMMA 2.2. *Let X be a Borel-dense subset of $\mathbf{R}$ and $f\colon\mathbf{R}\to\mathbf{R}$ an essential measurable function. There exists a subset A of $X\cap\operatorname{supp}(f)$ of cardinality c such that f acts injectively on A.*

PROOF. Let $S=\operatorname{supp}(f)=\{x\in\mathbf{R}\colon x\neq f(x)\}$, f' be the restriction of f to S, and $T=f(S)=f'(S)$. Thus T is uncountable and analytic. Let T^* be the set of all $t\in T$ for which $f'^{-1}(\{t\})$ is uncountable. Then T^* is analytic (see [**8**, p. 496]). If T^* is uncountable, choose for each $t\in T^*$ an element $x_t\in X\cap f'^{-1}(\{t\})$ and put $A=\{x_t\colon t\in T^*\}$. Then $|A|=|T^*|=c$. On the other hand, if T^* is countable, $T\setminus T^*$ is uncountable and analytic and hence $|X\cap f'^{-1}(T\setminus T^*)|=c$. As $f'^{-1}(\{t\})$ is countable for each $t\in T\setminus T^*$, we can select a subset A of $X\cap f'^{-1}(T\setminus T^*)$ of cardinality c on which f acts injectively. In either case, the result follows.

LEMMA 2.3. *Let X be Borel-dense of order 2 in $\mathbf{R}$ and R any nonreticulate member of $\mathscr{B}_{\mathbf{R}\times\mathbf{R}}$. Then $|(X\times X)\cap R|=c$.*

PROOF. It suffices to show that there is a system $\{R_i\colon i<c\}$ of c pairwise disjoint subsets R_i of R, each being a nonreticulate member of $\mathscr{B}_{\mathbf{R}\times\mathbf{R}}$. Let π_1 (π_2) be the canonical projection of R onto its first (second) coordinate, respectively. Let $A_i=\{x\in\mathbf{R}\colon \pi_i^{-1}(\{x\})\}$ is uncountable; then A_i is analytic ($i=1,2$). If A_1 is uncountable, we can choose a system $\{B_i\colon i<c\}$ of c pairwise disjoint subsets B_i of cardinality c of A_1 with $B_i\in\mathscr{B}_{\mathbf{R}}$. Put $R_i=\pi_1^{-1}(B_i)$ for each $i<c$ to obtain the result. Therefore let us assume that A_1 and A_2 are both countable. Put $R^*=R\setminus((A_1\times\mathbf{R})\cup(\mathbf{R}\times A_2))$. As $\pi_1(R^*)$ is uncountable and analytic, we can now choose c pairwise disjoint subsets B_i ($i<c$) of cardinality c of $\pi_1(R^*)$ each belonging to $\mathscr{B}_{\mathbf{R}}$. Again put $R_i=\pi_1^{-1}(B_i)$ ($i<c$) to obtain the result.

If $q\in A\times B\times C\times D$ is a quadruple and, for instance, $b\in B$, we write $q=(*,b,*,*)$ to denote that $q=(a,b,c,d)$ for some $a\in A$, $c\in C$, $d\in D$. Now we show:

THEOREM 2.4. *Let X be any Borel-dense Blackwell subset of $\mathbf{R}$. There exists a decomposition $X=\bigcup_{i<c}X_i$ of X into c pairwise disjoint subsets X_i with the following properties:*

(1) *Each subset X_i ($i<c$) is Borel-dense of order 2 in $\mathbf{R}$.*

(2) *Whenever* $i, j < c$, $X_i \subseteq Y \subseteq X$, *and* $f\colon Y \to \mathbf{R}$ *is an essential measurable function, then* $|f(X_i) \cap (X_j \cup (\mathbf{R} \setminus X))| = c$.

PROOF. Let $\{q_\alpha\colon \alpha < c\}$ be a list of all quadruples $q = (g, i, j, R)$, where $g\colon \mathbf{R} \to \mathbf{R}$ is an essential measurable function, the ordinals i, j are less than c, and R is a nonreticulate member of $\mathscr{B}_{\mathbf{R}\times\mathbf{R}}$. We also write $q_\alpha = (g_\alpha, i_\alpha, j_\alpha, R_\alpha)$. By Lemma 2.2, choose for each $\alpha < c$ a subset A_α of $X \cap \operatorname{supp}(g_\alpha)$ of cardinality c on which g_α acts injectively. We now choose elements $x_\alpha, u_\alpha, v_\alpha \in \mathbf{R}$ $(\alpha < c)$ inductively as follows. Assume that $\alpha < c$ and for each $\beta < \alpha$ we have found elements $x_\beta \in A_\beta$ and $u_\beta, v_\beta \in X$ such that $(u_\beta, v_\beta) \in R_\beta$ and the elements

$$(*) \qquad x_\beta, g_\beta(x_\beta), u_\beta, v_\beta \qquad (\beta < \alpha)$$

are all different from each other except that possibly $u_\beta = v_\beta$ for some $\beta < \alpha$. As $|A_\alpha| = c$, there exists $x_\alpha \in A_\alpha$ such that x_α and $g_\alpha(x_\alpha)$ are different from all elements listed in $(*)$. Next, as $|R_\alpha \cap (X \times X)| = c$ by Lemmas 2.1 and 2.3, we can choose $u_\alpha, v_\alpha \in X$ different from $x_\alpha, g_\alpha(x_\alpha)$, and all elements of $(*)$ such that $(u_\alpha, v_\alpha) \in R_\alpha$.

Now put

$$X_i' = \{x_\alpha, u_\alpha, v_\alpha\colon q_\alpha = (*, i, *, *),\ \alpha < c\} \\ \cup (X \cap \{g_\alpha(x_\alpha)\colon q_\alpha = (g_\alpha, *, i, *),\ \alpha < c\})$$

for each $i < c$. By construction, these sets X_i' are pairwise disjoint. Let $Z = X \setminus \bigcup_{i<c} X_i'$ and put $X_0 = X_0' \cup Z$, $X_i = X_i'$ for each $0 < i < c$, where $0 = \min\{i\colon i < c\}$. Then clearly condition (1) is satisfied.

Now let $f\colon Y \to \mathbf{R}$ be any essential measurable mapping where $X_i \subseteq Y \subseteq X$, and let $i, j < c$. We can extend f to a measurable function g defined on all of $\mathbf{R}$ (cf. [**8**, p. 434]). Then g is essential, and for any nonreticulate set R in $\mathscr{B}_{\mathbf{R}\times\mathbf{R}}$ there exists $\alpha = \alpha(R) < c$ with $(g, i, j, R) = q_\alpha$. Thus $x_\alpha \in X_i$ and $g(x_\alpha) \in X_j \cup (\mathbf{R} \setminus X)$. Since there are precisely c such nonreticulates R, we obtain $|f(X_i) \cap (X_j \cup (\mathbf{R} \setminus X))| = c$.

Note that each superset (in $\mathbf{R}$) of any of the sets X_i of Theorem 2.4 is a strong Blackwell set by Lemma 2.1. Now we give the

PROOF OF THEOREM 1. Choose a decomposition $X = \bigcup_{i\in\mathbf{R}} X_i$ of X into c pairwise disjoint subsets X_i with all the properties of Theorem 2.4. Let $\mathscr{F}$ be the system of all unions $U_A = \bigcup_{i\in A} X_i$ where $A \subseteq \mathbf{R}$. Now suppose that A, B are nonempty proper subsets of $\mathbf{R}$ and $f\colon U_A \to \mathbf{R}$ is an essential measurable function. Then $f(U_A)$ intersects $U_{\mathbf{R}\setminus B} \cap (\mathbf{R} \setminus X)$. Observing Lemma 2.1, it follows that $\mathscr{F}$ satisfies all the requirements of the theorem.

The following result, which is immediate both by Theorem 2.4 (and Lemma 2.1) and Theorem 1, generalizes Bhaskara Rao and Rao [**1**, Proposition 12] and Rosenstein [**12**, Theorem 9.6].

COROLLARY 2.5. *Let* X *be any Borel-dense Blackwell subset of* $\mathbf{R}$ *and* κ *any finite or infinite cardinal number with* $2 \leq \kappa \leq c$. *Then there exists a decomposition* $X = \bigcup_{i<\kappa} X_i$ *of* X *into* κ *pairwise disjoint nonanalytic super-rigid strong Blackwell subsets* X_i *such that, for any* $i, j < \kappa$ *with* $i \neq j$, *there is no measurable mapping of* X_i *into* X_j *with uncountable range.*

We also note the following related result.

COROLLARY 2.6. *Let X be any Borel-dense Blackwell subset of $\mathbf{R}$. There exists a super-rigid family $\mathscr{A}$, closed under complementation in X, of 2^c (nonanalytic) strong Blackwell subsets of X such that whenever $A, B \in \mathscr{A}$ with $A \neq B$, there exists no measurable mapping $f: A \to B$ which is either injective or surjective.*

PROOF. First choose a subfield $\mathscr{F}$ of $\mathscr{P}(X)$ with the properties (1)–(3) of Theorem 1, and let φ be an isomorphism from $\mathscr{P}(\mathbf{R})$ onto $\mathscr{F}$. Next split $\mathbf{R} = R_1 \cup R_2$ into two disjoint subsets of equal cardinality, and let $f: R_1 \to R_2$ be a bijection. For each subset $A \subseteq R_1$ put $S_A = A \cup (R_2 \setminus f(A))$. Then $\mathscr{S} = \{S_A : A \subseteq R_1\}$ is closed under complementation, has size 2^c, and consists only of pairwise incomparable elements. Now put $\mathscr{A} = \varphi(\mathscr{S})$ to obtain the result.

In view of Corollary 2.5, we wish to show that X is also the union of a well ordered chain of length c of pairwise nonisomorphic strong Blackwell subsets. In fact we have:

COROLLARY 2.7. *Let X be any Borel-dense Blackwell subset of $\mathbf{R}$, and let $(I, \leq)$ be any linearly ordered set with $|I| \leq c$ and without a greatest or a smallest element. Then there exists a super-rigid family $\mathscr{C}$ of strong Blackwell subsets of $\mathbf{R}$ such that:*

(1) *$(\mathscr{C}, \subseteq)$ is a chain isomorphic to $(I, \leq)$.*

(2) *Whenever $A, B \in \mathscr{C}$ with $A \subsetneq B$, there is no measurable injection of B into A, and also no measurable surjection of A onto B.*

(3) *$X = \bigcup\{A : A \in \mathscr{C}\}$ and $\bigcap\{A : A \in \mathscr{C}\} = \varnothing$.*

PROOF. Split $\mathbf{R} = \bigcup_{i \in I} X_i$ into $|I|$ pairwise disjoint nonempty subsets X_i. For each $i \in I$ let $A_i = \bigcup\{X_j : j \in I,\ j \leq i\}$, and put $\mathscr{A} = \{A_i : i \in I\}$. Clearly $(I, \leq)$ and $(\mathscr{A}, \subseteq)$ are order-isomorphic. Now let $\mathscr{C}$ be the image of $\mathscr{A}$ under any isomorphism from $\mathscr{P}(\mathbf{R})$ onto $\mathscr{F}$, where $\mathscr{F}$ is the subfield of $\mathscr{P}(X)$ of Theorem 1. The result follows.

Let us say that two sets of reals have *incomparable continuity type*, if none of them is a continuous image of the other (cf. [**8**, p. 428]). The following observation is in the spirit of a result of Sierpiński [**17**] (cf. [**12**, Theorem 9.10(3)]) dealing with order types of subsets of $\mathbf{R}$.

COROLLARY 2.8. *There are c Borel-dense subsets of $\mathbf{R}$ any two of which have incomparable continuity types but differ by only two elements.*

PROOF. Let X, Y be two of the subsets X_i of $\mathbf{R}$ of Theorem 2.4. We claim that the sets $X_y = X \cup \{y\} (y \in Y)$ satisfy our requirements. Suppose that $y, z \in Y$ with $y \neq z$ and $f: X_y \to X_z$ is continuous and onto. Let $x \in X_y$ with $f(x) = z$. There is a neighborhood U of x in X_y which is disjoint with $f(U)$. Hence $U \setminus \{y\} \subseteq f(\text{supp}(f))$. As X is Borel-dense, U is uncountable and f is essential. But then $|X_z \cap Y| = c$ by Theorem 2.4, a contradiction.

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