HYPONORMALITY OF TOEPLITZ OPERATORS

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ABSTRACT. For φ in $L^\infty(\partial\mathbf{D})$, let $\varphi = f + \bar{g}$ where f and g are in H^2. In this note, it is shown that the Toeplitz operator T_φ is hyponormal if and only if $g = c + T_{\bar{h}}f$ for some constant c and some function h in $H^\infty(\mathbf{D})$ with $\|h\|_\infty \le 1$.

For φ in $L^\infty(\partial\mathbf{D})$, the Toeplitz operator T_φ is the operator on H^2 of the unit disk $\mathbf{D}$ given by $T_\varphi u = P\varphi u$ where P is the orthogonal projection of $L^2(\partial\mathbf{D})$ onto H^2. An operator A is called hypnormal if its self-commutator $A^*A - AA^*$ is positive. The goal of this paper is to characterize hyponormal Toeplitz operators.

Brown and Halmos began the systematic study of the algebraic properties of Toeplitz operators and showed, [3, p. 98], that T_φ is normal if and only if $\varphi = \alpha + \beta\rho$ where α and β are complex numbers and ρ is a real valued function in L^∞. There are many results concerning hyponormality of Toeplitz operators in the literature and properties of hyponormal Toeplitz operators have played an important role in work on Halmos's Problem 5, [7], "Is every subnormal Toeplitz operator either normal or analytic?" but a characterization has been lacking. (For references, see the bibliography; [6] surveys much of the literature.)

THEOREM 1. *If φ is in $L^\infty(\partial\mathbf{D})$, where $\varphi = f + \bar{g}$ for f and g in H^2, then T_φ is hyponormal if and only if*

$$g = c + T_{\bar{h}}f$$

for some constant c and some function h in $H^\infty(\mathbf{D})$ with $\|h\|_\infty \le 1$.

The basis of the proof is a dilation theorem; we will use the notation and formulation of Sarason [13, Theorem 1]. The unilateral (forward) shift on H^2 will be denoted by U. Moreover, the proof uses standard results about Hankel operators, for example, see [12]. For ψ in L^∞, the Hankel operator H_ψ is the operator on H^2 given by

$$H_\psi u = J(I - P)(\psi u)$$

where J is the unitary operator from $H^{2\perp}$ onto H^2

$$J(e^{-in\theta}) = e^{i(n-1)\theta}.$$

Denoting by v^* the function $v^*(e^{i\theta}) = \overline{v(e^{-i\theta})}$, another way to put this is that H_ψ is the operator on H^2 defined by

$$(1) \qquad \langle zuv, \overline{\psi}\rangle = \langle H_\psi u, v^*\rangle, \quad \text{for all } v \in H^\infty.$$

Necessary facts about Hankel operators include

- $H_{\psi_1} = H_{\psi_2}$ if and only if $(I - P)\psi_1 = (I - P)\psi_2$.
- $\|H_\psi\| = \inf\{\|\varphi\|_\infty \colon (I - P)\psi = (I - P)\varphi\}$.
- $H_\psi^* = H_{\psi^*}$.
- Either H_ψ is one-to-one or $\ker(H_\psi) = \chi H^2$ where χ is an inner function. The closure of the range of H_ψ is H^2 in the former case and $(\chi^* H^2)^\perp$ in the latter.
- $H_\psi U = U^* H_\psi$.

PROOF. Let $\varphi = f + \bar{g}$ where f and g are in H^2.

The first step of the proof is one of the equivalences of Proposition 11 of [6]. For every polyomial p in H^2,

$$
\begin{aligned}
\langle (T_\varphi^* T_\varphi - T_\varphi T_\varphi^*)(p), p \rangle &= \langle T_\varphi p, T_\varphi p \rangle - \langle T_\varphi^* p, T_\varphi^* p \rangle \\
&= \langle fp + P\bar{g}p, fp + P\bar{g}p \rangle - \langle P\bar{f}p + gp, P\bar{f}p + gp \rangle \\
&= \langle \bar{f}p, \bar{f}p \rangle - \langle P\bar{f}p, P\bar{f}p \rangle - \langle \bar{g}p, \bar{g}p \rangle + \langle P\bar{g}p, P\bar{g}p \rangle \\
&= \langle \bar{f}p, (I - P)\bar{f}p \rangle - \langle \bar{g}p, (I - P)\bar{g}p \rangle \\
&= \langle (I - P)\bar{f}p, (I - P)\bar{f}p \rangle - \langle (I - P)\bar{g}p, (I - P)\bar{g}p \rangle \\
&= \|H_{\bar{f}}p\|^2 - \|H_{\bar{g}}p\|^2.
\end{aligned}
$$

Since the polynomials are dense in H^2 and since the Hankel and Toeplitz operators involved are bounded, we see that T_φ is hyponormal if and only if for all u in H^2,

$$
(2) \qquad\qquad\qquad\qquad \|H_{\bar{g}}u\| \le \|H_{\bar{f}}u\|.
$$

Let K denote the closure of the range of $H_{\bar{f}}$, and let S denote the compression of U to K. Since K is invariant for U^*, the operator S^* is the restriction of U^* to K.

Suppose first that T_φ is hyponormal. Define an operator A on the range of $H_{\bar{f}}$ by

$$
A(H_{\bar{f}}u) = H_{\bar{g}}u.
$$

If $H_{\bar{f}}u_1 = H_{\bar{f}}u_2$, so that $H_{\bar{f}}(u_1 - u_2) = 0$, then the inequality (2) implies that $H_{\bar{g}}(u_1 - u_2) = 0$ too and it follows that A is well defined. Moreover, inequality (2) implies $\|A\| \le 1$ so A has an extension to K, which will also be denoted A, with the same norm.

Now by the intertwining formula for Hankel operators and the fact that K is invariant for U^*, we have

$$
H_{\bar{g}}U = AH_{\bar{f}}U = AU^* H_{\bar{f}} = AS^* H_{\bar{f}}
$$

and also

$$
H_{\bar{g}}U = U^* H_{\bar{g}} = U^* AH_{\bar{f}} = S^* AH_{\bar{f}}.
$$

Since the range of $H_{\bar{f}}$ is dense in K, we find that $AS^* = S^* A$ on K, or taking adjoints, that

$$
SA^* = A^* S.
$$

By [13, Theorem 1] (or by the usual theory of the unilateral shift if $K = H^2$), there is a function k in $H^\infty(\mathbf{D})$ with $\|k\|_\infty = \|A^*\| = \|A\|$ such that A^* is the

compression to K of T_k. Since K is invariant for $T_k^* = T_{\bar{k}}$, this means that A is the restriction of $T_{\bar{k}}$ to K and

$$(3) \qquad\qquad H_{\bar{g}} = T_{\bar{k}} H_{\bar{f}}.$$

Conversely, if equation (3) holds for some k in $H^\infty(\mathbf{D})$ with $\|k\|_\infty \le 1$, then clearly inequality (2) holds for all u, and T_φ is hyponormal.

The proof will be completed by analyzing the relationship given by equation (3). Using the formulation (1), equation (3) holds if and only if for all H^∞ functions u, v,

$$\langle zuv, g \rangle = \langle H_{\bar{g}} u, v^* \rangle = \langle T_{\bar{k}} H_{\bar{f}} u, v^* \rangle = \langle H_{\bar{f}} u, k v^* \rangle$$
$$= \langle zuk^* v, f \rangle = \langle zuv, \overline{k^*} f \rangle = \langle zuv, T_{\overline{k^*}} f \rangle.$$

Since the closed span of $\{zuv : u, v \in H^\infty\}$ is zH^2 this means that equation (3) holds if and only if $g = c + T_{\bar{h}} f$ for $h = k^*$. (Note that $\|k\|_\infty = \|k^*\|_\infty$.) □

In the cases for which T_φ is normal, h is a constant of modulus 1 and in the cases for which T_φ is known to be subnormal but not normal, h is a constant of modulus less than 1.

It is of some interest to investigate the uniqueness of the functions h that relate f and g. Suppose h_1 and h_2 are in H^∞ and $c_1 + T_{\overline{h_1}} f = g = c_2 + T_{\overline{h_2}} f$. This is possible if and only if

$$T_{\bar{z}} T_{\overline{h_1}} f = T_{\bar{z}} T_{\overline{h_2}} f,$$

that is, if and only if $T_{\overline{zh_1 - zh_2}} f = 0$. Thus, f must be in $(z\chi H^2)^\perp$ where χ is the inner factor of $h_1 - h_2$. If f is not in any such subspace, the corresponding function h must be unique for every g. On the other hand, if χ is an inner function such that f is in $(z\chi H^2)^\perp$ and $c_1 + T_{\overline{h_1}} f = g$, then for any h_3 in H^∞ and $h_2 = h_1 + z\chi h_3$, it follows that $g = c_2 + T_{\overline{h_2}} f$ for some constant c_2.

In [6], the author made the following generalization of the set of g in H^2 for which $T_{f+\bar{g}}$ is hyponormal.

DEFINITION. Let $\mathscr{K} = \{v \in H^\infty : v(0) = 0 \text{ and } \|v\|_2 \le 1\}$. For f in H^2, let G_f denote the set of g in H^2 such that for every u in H^2,

$$\sup_{v_0 \in \mathscr{K}} |\langle uv_0, g \rangle| \le \sup_{v_0 \in \mathscr{K}} |\langle uv_0, f \rangle|.$$

To see how this definition is relevant to our work, note that if f is in H^∞ and u is in H^2, then by equation (1),

$$\sup_{v_0 \in \mathscr{K}} |\langle uv_0, f \rangle| = \|H_{\bar{f}} u\|.$$

Thus, when f and g are bounded analytic, $T_{f+\bar{g}}$ is hyponormal if and only if g is in G_f.

For f in H^2, not necessarily the analytic part of a function in L^∞, if we regard $H_{\bar{f}}$ as a bounded operator from H^∞ into H^2, then we may proceed exactly as above to prove the following theorem.

THEOREM 2. *If f and g are in H^2, then g is in G_f if and only if $g = c + T_{\bar{h}} f$, for some constant c and some function h in $H^\infty(\mathbf{D})$ with $\|h\|_\infty \le 1$.*

We can now easily answer Question 1 of [6].

COROLLARY 3. *For f in H^2, the following hold.*

(1) *f is in G_f.*

(2) *If g is in G_f, then $g + \lambda$ is in G_f for all complex numbers λ.*

(3) *G_f is balanced and convex; that is, if g_1 and g_2 are in G_f and $|s_1| + |s_2| \leq 1$, then $s_1 g_1 + s_2 g_2$ is also in G_f.*

(4) *G_f is weakly closed.*

(5) *$T_{\bar{\chi}} G_f \subset G_f$ for every inner function χ.*

Conversely, if G is a set that satisfies properties (1) to (5), then $G \supset G_f$.

PROOF. That G_f has the indicated properties is Theorem 12 of [6].

To prove the converse statement, note that f is in G and by (3), (4), and (5), G contains $T_{\bar{h}} f$ whenever h is in the weakly closed convex hull of the set of inner functions. By a theorem of Marshall [11, Corollary, p. 496], the norm closed convex hull of the Blaschke products in H^∞ is the unit ball of H^∞. Property (2) and Theorem 2 now imply the desired inclusion. □

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