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DUAL COMPLEMENTORS ON BANACH ALGEBRAS

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ABSTRACT. We study semisimple annihilator Banach algebras A with a right complementor p such that the mapping q on the closed left ideals J of A given by $J^q = l_A([r_A(J)]^p)$ is a left complementor on A. A right complementor p with this property is called dual, and the pair (p, q) is called a dual pair of complementors on A.

1. Introduction. An annihilator B^*-algebra A has a dual right complementor p defined on it. It is given by $I^p = [l_A(I)]^*$, for all closed right ideals I of A. In §3 we obtain some properties of dual right complementors and dual pairs of complementors. In §4 we consider the existence of dual right complementors. We show that if A is a semisimple annihilator Banach algebra which is a dense ideal in a B^*-algebra B, then there exists a dual right complementor on A. In particular A is a dual algebra (Corollary 4.3). Using this fact we obtain an extension of Kaplansky's result [**6**, Theorem 2.1, p. 222]. We also show that if H is an infinite dimensional Hilbert space and A is a semisimple annihilator Banach algebra which is a dense ideal in $LC(H)$, the algebra of all compact linear operators on H, then every right complementor on A is dual.

2. Preliminaries. Let A be a Banach algebra. By an ideal we shall always mean a two-sided ideal unless otherwise specified. For any set S in A, $l_A(S)$ and $r_A(S)$ will denote, respectively, the left and right annihilators of S in A, and $\mathrm{cl}_A(S)$ will denote the closure of S in A. The socle of A will be denoted by S_A. We call A an *annihilator algebra* if $l_A(A) = r_A(A) = (0)$, and if for every proper closed right ideal I and every proper closed left ideal J of A, $l_A(I) \neq (0)$ and $r_A(J) \neq (0)$. If, in addition, $r_A(l_A(I)) = I$ and $l_A(r_A(J)) = J$, then A is called a *dual algebra*. A B^*-algebra with dense socle is a dual algebra [**6**, Theorem 2.1, p. 222]. (See also [**4** and **8**].)

All Banach algebras considered in this paper are over the complex field.

Let A be a Banach algebra and let L_r be the set of all closed right ideals of A. We say that A is *right complemented* (r.c.) if there exists a mapping $p\colon R \to R^p$ of L_r into itself having the following properties:

(C_1) $I \cap I^p = (0)$ $(I \in L_r)$;

(C_2) $I + I^p = A$ $(I \in L_r)$;

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(C_3) $(I^p)^p = I$ $(I \in L_r)$;

(C_4) if $I_1 \subseteq I_2$ then $I_2^p \subseteq I_1^p$ $(I_1, I_2 \in L_r)$.

The mapping p is called a *right complementor* on A. Analogously, we define a left complemented (l.c.) Banach algebra and a left complementor. A Banach algebra that is both left and right complemented is said to be *bicomplemented*. We will denote the set of all closed left ideals in A by L_l.

In what follows, $p(q)$ will always denote a right (left) complementor on a Banach algebra A, unless mentioned otherwise.

Let A be an r.c. Banach algebra with a right complementor p. An idempotent e in A is called a *p-projection* if $(eA)^p = (1-e)A$. If, moreover, e is a minimal idempotent, e is called a *minimal p-projection*. If A is a semisimple annihilator r.c. Banach algebra, then every nonzero closed right ideal of A contains a minimal p-projection [**9**, p. 654].

3. Dual complementors and dual pairs of complementors. Let A be a semisimple annihilator Banach algebra and let p be a right complementor on A. Let u be the mapping on L_l given by

$$J^u = l_A([r_A(J)]^p) \qquad (J \in L_l).$$

We say that u is *derived from p*. We call p a *dual right complementor* whenever u is a left complementor on A.

Similarly if q is a left complementor on A then the mapping v on L_r given by

$$I^v = r_A([l_A(I)]^q) \qquad (I \in L_r)$$

is said to be *derived from q*. We call q a *dual left complementor* whenever v is a right complementor on A.

REMARK. We recall that a semisimple annihilator bicomplemented Banach algebra is a dual algebra [**1**, Corollary, p. 45]. We will use this fact throughout the paper.

PROPOSITION 3.1. *Let p and q be, respectively, right and left complementors on A. Then q is derived from p if and only if p is derived from q.*

PROOF. Suppose that q is derived from p. Let $I \in L_r$ and $J = l_A(I)$. By the duality of A, $I = r_A(J)$ so that

$$I^p = [r_A(J)]^p \quad \text{and} \quad l_A(I^p) = l_A([r_A(J)]^p) = J^q.$$

Therefore

$$I^p = r_A(l_A(I^p)) = r_A(J^q) = r_A([l_A(I)]^q)$$

which shows that p is derived from q.

The converse implication is proved in the same way.

DEFINITION. A pair (p, q) is called a dual pair of complementors on A if q is derived from p (or equivalently p is derived from q). Thus both p and q are dual complementors, each derived from the other.

EXAMPLE. Let A be an annihilator B^*-algebra. Then the mapping $p\colon I \to I^p = [l_A(I)]^*$ on L_r is a right complementor on A [**9**, p. 652]. Similarly the mapping $q\colon J \to J^q = [r_A(J)]^*$ on L_l is a left complementor on A. Using the duality of A, we have

$$J^q = l_A(r_A([r_A(J)]^*)) = l_A([l_A(r_A(J))]^*) = l_A([r_A(J)]^p)$$

for all $J \in L_l$. Therefore q is derived from p so that p is dual and the pair (p, q) is a dual pair of complementors on A.

THEOREM 3.2. *Let A be a semisimple annihilator bicomplemented Banach algebra with complementors p and q. Then the following statements are equivalent:*
(i) *The pair (p, q) is a dual pair of complementors on A.*
(ii) $[r_A(J)]^p = r_A(J^q)$ *for every closed left ideal J of A.*
(iii) $[l_A(I)]^q = l_A(I^p)$ *for every closed right ideal I of A.*

PROOF. (i)→(ii). Assume (i). Then $J^q = l_A([r_A(J)]^p)$ for all $J \in L_l$. Therefore, by the duality of A,

$$r_A(J^q) = r_A(l_A([r_A(J)]^p)) = [r_A(J)]^p,$$

which is (ii).
(ii)→(iii). Assume (ii). Let $I \in L_r$ and let $J = l_A(I)$. Then, $I = r_A(J)$ and

$$[l_A(I)]^q = [l_A(r_A(J))]^q = J^q.$$

But, by (ii),

$$J^q = l_A([r_A(J)]^p) = l_A(I^p).$$

Hence (iii).
(iii)→(i). Assume (iii). Let $J \in L_l$ and $I = r_A(J)$. Then, by (iii),

$$J^q = [l_A(I)]^q = l_A(I^p) = l_A([r_A(J)]^p)$$

which shows that q is derived from p and therefore (p, q) is a dual pair of complementors on A.

PROPOSITION 3.3. *Let (p, q) be a dual pair of complementors on a semisimple annihilator Banach algebra A. Then an idempotent e in A is a p-projection if and only if it is a q-projection.*

PROOF. Suppose that e is a p-projection, i.e., $(eA)^p = (1 - e)A$. Then

$$\begin{aligned}(Ae)^q &= l_A([r_A(Ae)]^p) = l_A([(1-e)A]^p) \\ &= l_A(eA) = A(1-e)\end{aligned}$$

which shows that e is also a q-projection. The converse is proved similarly.

COROLLARY 3.4. *Let A be a semisimple annihilator Banach algebra with a dual right complementor p. Let $\{e_\alpha : \alpha \in \Omega\}$ be a maximal orthogonal family of minimal p-projections in A. Then every $x \in A$ can be expressed as*

$$x = \sum_\alpha e_\alpha x = \sum_\alpha x e_\alpha,$$

where convergence is with respect to the net of finite partial sums.

PROOF. Let q be the left complementor on A derived from p. Since (p, q) is a dual pair of complementors on A, by Proposition 3.3, $\{e_\alpha : \alpha \in \Omega\}$ is also a maximal orthogonal family of minimal q-projections in A. The conclusion now follows from [**12**, Theorem 5.9, p. 268].

THEOREM 3.5. *Let A be a semisimple annihilator Banach algebra and let p be a right complementor on A. Then p is dual if and only if* (i) *A is a dual algebra and* (ii) $l_A(I) + l_A(I^p) = A$ *for every closed right ideal I of A.*

PROOF. Suppose that p is dual and that q is the left complementor on A derived from p. Since A is bicomplemented, A is dual. Now let $I \in L_r$ and $J = l_A(I)$. Then

$$J^q = l_A([r_A(J)]^p) = l_A([r_A(l_A(I))]^p) = l_A(I^p).$$

Hence

$$l_A(I) + l_A(I^p) = J + J^q = A.$$

Suppose conversely that properties (i) and (ii) hold. Since A is semisimple, the sum $l_A(I) + l_A(I^p) = A$ is direct. To show that p is dual, we must verify that the mapping q on L_l given by

$$J^q = l_A([r_A(J)]^p) \qquad (J \in L_l)$$

is a left complementor on A. It is easy to see that for $J \in L_l$, we have $J \cap J^q = (0)$ and $J + J^q = A$. In fact, if we let $I = r_A(J)$ then $J = l_A(I)$, $J^q = l_A(I^p)$ and the sum in (ii) is direct. Moreover,

$$\begin{aligned}(J^q)^q &= [l_A([r_A(J)]^p)]^q = [l_A(I^p)]^q \\ &= l_A([r_A(l_A(I^p))]^p) = l_A((I^p)^p) \\ &= l_A(I) = l_A(r_A(J)) = J.\end{aligned}$$

It remains to show that if $J_1 \subset J_2$ then $J_2^q \subset J_1^q$, for all $J_1, J_2 \in L_l$. We observe that the duality of A implies that $r_A(J_2) \subset r_A(J_1)$ so that $[r_A(J_1)]^p \subset [r_A(J_2)]^p$ and therefore

$$J_2^q = l_A([r_A(J_2)]^p) \subset l_A([r_A(J_1)]^p) = J_1^q.$$

Thus q is a left complementor on A and this completes the proof.

Let A be an annihilator A^*-algebra with a right complementor p. Since $I^* + (I^p)^* = A$, for $I \in L_r$, and since every $J \in L_l$ can be written as $J = I^*$, for some $I \in L_r$, it follows that the mapping q on L_l given by

(1) $$J^q = (I^p)^*, \quad \text{where } J = I^*,$$

is a left complementor on A. Thus, in particular, A is a dual algebra. The following theorem shows when (p, q) is a dual pair of complementors on A, i.e., q is derived from p.

THEOREM 3.6. *Let A be an annihilator A^*-algebra with a right complementor p, and let q be the left complementor on A given in* (1). *Then (p, q) is a dual pair of complementors on A if and only if*

(2) $$[(l_A(I))^*]^p = [l_A(I^p)]^*$$

for every closed right ideal I of A.

PROOF. Suppose that (2) holds. Let $J \in L_l$ and $I = J^* \in L_r$. Then $J^q = (I^p)^*$ and

$$\begin{aligned}[r_A(J)]^p &= [r_A(I^*)]^p = [(l_A(I))^*]^p \\ &= [l_A(I^p)]^* = r_A([I^p]^*) = r_A(J^q).\end{aligned}$$

Therefore, by Theorem 3.2, (p,q) is a dual pair of complementors on A.

Suppose conversely that (p,q) is a dual pair of complementors on A. Let $I \in L_r$ and $J = I^* \in L_l$. Then

$$[(l_A(I))^*]^p = [r_A(I^*)]^p = [r_A(J)]^p$$

and

$$[l_A(I^p)]^* = r_A([I^p]^*) = r_A(J^q).$$

An application of Theorem 3.2 completes the proof.

The right complementor p on an annihilator B^*-algebra A given by $I^p = [l_A(I)]^*$ has property (2). In fact,

$$l_A(I^p) = l_A([l_A(I)]^*) = l_A(r_A(I^*)) = I^*$$

and

$$[(l_A(I))^*]^p = [r_A(I^*)]^p = [l_A(r_A(I^*))]^* = I,$$

whence (2). We have

$$J^q = (I^p)^* = l_A(I) = [r_A(I^*)]^* = [r_A(J)]^*.$$

4. Existence of dual complementors.

Notation. Let A and B be Banach algebras such that A is an ideal of B. Let $B \cdot A$ (resp. $A \cdot B$) be the subspace of A generated by the elements ba (resp. ab), $b \in B$ and $a \in A$. Clearly $B \cdot A$ and $A \cdot B$ are ideals of A. We will denote the set of all closed left (resp. right) ideals in B by $\mathscr{L}_l$ (resp. $\mathscr{L}_r$). As before L_l (resp. L_r) will denote the set of all closed left (resp. right) ideals in A.

LEMMA 4.1. *Let A and B be semisimple Banach algebras such that A is a dense ideal in B. Assume that B has a bounded approximate identity contained in A and that $B \cdot A$ and $A \cdot B$ are dense in A. Then the mapping $I \to \mathrm{cl}_B(I)$ (resp. $J \to \mathrm{cl}_B(J)$) is bijective from the set of all closed right (resp. left) ideals of A onto the set of all closed right (resp. left) ideals of B. The inverse mapping is $R \to R \cap A$ (resp. $N \to N \cap A$) on the closed right (resp. left) ideals of B.*

PROOF. Let $\{u_\alpha\}$ be a bounded approximate identity of B contained in A. Since $B \cdot A$ and $A \cdot B$ are dense in A, $\{u_\alpha\}$ is also an approximate identity of A by [**3**, Proposition 3.3, p. 5]. Thus both A and B contain approximate left and right units. The conclusion of the lemma now follows from [**7**, Theorem 2.3, p. 299].

THEOREM 4.2. *Let A be a semisimple annihilator Banach algebra which is a dense ideal in a B^*-algebra B. Then there exists a dual right complementor on A.*

PROOF. Since $S_A \subset S_B$ and S_A is dense in B, it follows that B is an annihilator algebra, $B \cdot A$ and $A \cdot B$ are dense in A. Moreover, by [**5**, Proposition 1.7.2, p. 15], A contains a bounded approximate identity of B. Therefore, by [**12**, Theorem 5.2(ii), p. 265], the right complementor

$$r\colon R \to R^r = [l_B(R)]^* \qquad (R \in \mathscr{L}_r)$$

on B induces the right complementor

$$p\colon I \to I^p = [l_B(\mathrm{cl}_B(I))]^* \cap A = [l_B(I)]^* \cap A \qquad (I \in L_r)$$

on A. Similarly the left complementor

$$s\colon N \to N^s = [r_B(N)]^* \qquad (N \in \mathscr{L}_l)$$

on B induces the left complementor

$$q\colon J \to J^q = [r_B(\mathrm{cl}_B(J))]^* \cap A = [r_B(J)]^* \cap A \qquad (J \in L_l)$$

on A. Let $I \in L_r$. Then (using Lemma 4.1)

$$\begin{aligned}[l_A(I)]^q &= [r_B(l_B(I))]^* \cap A = l_B([l_B(I)]^*) \cap A \\ &= l_B(\mathrm{cl}_B([l_B(I)]^* \cap A)) \cap A = l_A([l_B(I)]^* \cap A) = l_A(I^p).\end{aligned}$$

Hence, by Theorem 3.2, (p, q) is a dual pair of complementors on A and so p is dual.

COROLLARY 4.3. *A semisimple annihilator Banach algebra A which is a dense ideal in a B^*-algebra is a dual algebra.*

PROOF. By Theorem 4.2, A is bicomplemented and therefore dual.

Corollary 4.3 is useful in proving the following theorem which gives an extension of Kaplansky's result [**6**, Theorem 2.1, p. 222].

THEOREM 4.4. *Let A be a Banach algebra which is a dense ideal in a B^*-algebra B. Then A is a dual algebra if and only if S_A is dense in A.*

PROOF. Assume that S_A is dense in A. Now A, as an ideal in a semisimple Banach algebra B, is also semisimple. By [**3**, Proposition 2.2, p. 3], the embedding of A in B is continuous. Hence S_A is dense in B. As $S_A \subset S_B$, B has dense socle and so B is a dual algebra. Since S_A is dense in A, Lemma 4.1 is in force here. Let $I \neq A$ be a closed right ideal of A. Then, by Lemma 4.1, $W = \mathrm{cl}_B(I)$ is a closed right ideal of B, $W \neq B$. Let $x_0 \in l_B(W)$, $x_0 \neq 0$. Then $Ax_0 \subset l_B(W) \cap A$ and $Ax_0 \neq (0)$. ($Ax_0 = (0)$ implies $Bx_0 = (0)$ and consequently $x_0 = 0$ since B is semisimple.) Hence $l_A(I) \neq (0)$. Similarly $r_A(J) \neq (0)$ for every closed left ideal J of A, $J \neq A$. Thus A is an annihilator algebra and therefore by Corollary 4.3, A is also a dual algebra.

The converse is well known (see [**8**, Corollary (2.8.16), p. 100]).

THEOREM 4.5. *Let H be an infinite dimensional Hilbert space and let A be a semisimple annihilator Banach algebra which is a dense ideal in $LC(H)$, the algebra of all compact linear operators on H. Then every right complementor p on A is dual.*

PROOF. For each $a \in A$, let L_a be the linear operator on A given by $L_a(x) = ax$, for all $x \in A$. Let $\|a\|'_A = \|L_a\|$, where $\|L_a\|$ is the operator bound norm of L_a on A. For convenience let $B = LC(H)$. Since B is a B^*-algebra (with norm $|\cdot|$) and A is a dense ideal in B, by [**10**, Proposition 2.2, p. 73], the norms $|\cdot|$ and $\|\cdot\|'_A$ are equivalent on A and, by [**5**, Proposition 1.7.2, p. 15], A contains a bounded approximate identity of B. Therefore, if p is a right complementor on A then, by [**12**, Theorem 5.2(i), p. 265], the mapping

$$r\colon R \to R^r = \mathrm{cl}_B([R \cap A]^p) \qquad (R \in \mathscr{L}_r)$$

is a right complementor on B. Since H is infinite dimensional, by [**2**, Theorem 6.8, p. 471], r is continuous and so, by [**2**, Theorem 6.11, p. 473], there exists an involution $*'$ and an equivalent norm $|\cdot|'$ on B under which B is a B^*-algebra and such that

$$R^r = [l_B(R)]^{*'} \qquad (R \in \mathscr{L}_r).$$

Let $I \in L_r$ and $R = \mathrm{cl}_B(I) \in \mathscr{L}_r$. Then, by Lemma 4.1, we have

$$R^r = \mathrm{cl}_B([R \cap A]^p) = \mathrm{cl}_B(I^p).$$

Therefore

$$\begin{aligned} I^p &= \mathrm{cl}_B(I^p) \cap A = R^r \cap A \\ &= [l_B(\mathrm{cl}_B(I))]^{*'} \cap A = [l_B(I)]^{*'} \cap A. \end{aligned}$$

We can now apply the argument in the proof of Theorem 4.2 (replacing $*$ by $*'$) to show that p is dual.

As our final observation, we will show that [**12**, Theorem 4.4, p. 263] carries over to semisimple annihilator r.c. Banach algebras with dual right complementors. We will need the following result.

PROPOSITION 4.6. *Let B be a semisimple annihilator r.c. Banach algebra with a dual right complementor p. Let I be a closed right ideal of B and let $\{e_\alpha : \alpha \in \Gamma\}$ be a maximal orthogonal family of minimal p-projections contained in I. Then*

(a) $I = \mathrm{cl}_B(\sum e_\alpha A : \alpha \in \Gamma)$.

(b) *If $\{e_\alpha : \alpha \in \Lambda\}$ is a maximal orthogonal family of minimal p-projections in A, where the index set Λ contains the index set Γ, then*

$$l_B(I) = \mathrm{cl}_B\left(\sum Ae_\alpha : \alpha \in \Lambda, \alpha \notin \Gamma\right).$$

PROOF. (a) This is contained in the proof of [**11**, Proposition 5.3, p. 7].

(b) Let q be the left complementor on B derived from p. By Proposition 3.3, each e_α, $\alpha \in \Lambda$, is a minimal q-projection and by (a), $e_\alpha \in l_B(I)$ for all $\alpha \notin \Gamma$. Let $J = l_B(I)$ and suppose that there exists a minimal q-projection e in J which is orthogonal to all e_α, $\alpha \in \Lambda$, $\alpha \notin \Gamma$, i.e., $ee_\alpha = e_\alpha e = 0$ for all $\alpha \in \Lambda$, $\alpha \notin \Gamma$. Since $JI = (0)$, it follows that $ee_\alpha = 0$ for all $\alpha \in \Lambda$. Therefore, by Corollary 3.4, $e = \sum_\alpha ee_\alpha = 0$; a contradiction. Hence $\{e_\alpha : \alpha \in \Lambda, \alpha \notin \Gamma\}$ is a maximal orthogonal family of minimal q-projections in J and so, by (a), rephrased for closed left ideals and q, we have $J = \mathrm{cl}_B(\sum Ae_\alpha : \alpha \in \Lambda, \alpha \notin \Gamma)$.

THEOREM 4.7. *Let B be a semisimple annihilator r.c. Banach algebra with a dual right complementor p. A dense subalgebra A which contains every minimal p-projection of B is a dual algebra.*

PROOF. This is the same as the proof of [**12**, Theorem 4.4, p. 263], we need only to replace there "selfadjoint minimal idempotent" by "minimal p-projection" and use Proposition 4.6. A is necessarily semisimple [**12**, p. 264].

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