PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 103, Number 3, July 1988

# THE ONE-RADIUS THEOREM IS NOT TRUE FOR BOUNDED REAL-ANALYTIC FUNCTIONS

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(Communicated by Irwin Kra)

*Dedicated to Professor Yukihiro Kodama on his 60th birthday*

ABSTRACT. There exists a bounded real-analytic function on the unit ball which does not satisfy Rudin's one-radius theorem.

Let $B$ be the open unit ball in $C^N$ and $S$ be the unit sphere in $C^N$. Put $\overline{B} = B \cup S$. Let $\sigma$ be the rotation-invariant positive Borel measure on $S$ with $\sigma(S) = 1$. We denote by $\operatorname{Aut}(B)$ the group of automorphisms in $B$. For a point $a$ in $B$, $\psi_a$ denotes the automorphism of $B$ defined by

$$\psi_a(z) = \frac{a - P_a z - s_a Q_a z}{1 - \langle z, a\rangle} \quad \text{for } z \in B,$$

where $P_a$ is the orthogonal projection of $C^N$ onto the subspace generated by $a$, $Q_a = I - P_a$, $s_a = (1-|a|^2)^{1/2}$ and $\langle z, a\rangle = \sum_{k=1}^N z_k \bar{a}_k$. For $f \in C^2(B)$ and $a \in B$, put

$$(\widetilde{\Delta} f)(a) = \Delta(f \circ \psi_a)(0) = 4\sum_{k=1}^{N} D_k \overline{D}_k (f \circ \psi_a)(0),$$

where $D_k = \partial/\partial z_k$ and $\overline{D}_k = \partial/\partial \bar{z}_k$. The operator $\widetilde{\Delta}$ is called the invariant Laplacian. A function $f$ in $C^2(B)$ with $\widetilde{\Delta} f = 0$ is called *$\mathscr{M}$-harmonic*. A function $f$ in $C(B)$, the space of continuous functions in $B$, is said to have the *invariant mean value property* if

$$f(\psi(0)) = \int_S f(\psi(r\varsigma))\, d\sigma(\varsigma)$$

for every $\psi \in \operatorname{Aut}(B)$ and $0 < r < 1$.

It is well known that every $\mathscr{M}$-harmonic function in $B$ has the invariant mean value property, and every $f \in C(B)$ with the invariant mean value property is $\mathscr{M}$-harmonic [**1**, p. 52]. Furthermore, Rudin proved that a much weaker mean value property implies $\mathscr{M}$-harmonicity as follows.

THE ONE-RADIUS THEOREM [**1**, p. 58]. *Suppose $u \in C(\overline{B})$, and suppose that there corresponds to every $z \in B$ just one radius $r(z)$ $(0 < r(z) < 1)$ such that*

$$u(z) = \int_S u(\psi_z(r(z)\varsigma))\, d\sigma(\varsigma).$$

*Then $u$ is $\mathscr{M}$-harmonic in $B$.*

Received by the editors March 16, 1987 and, in revised form June 24, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 31B05.
*Key words and phrases.* Real-analytic function, $\mathscr{M}$-harmonic, invariant mean value property.
The author was partly supported by Grant-in-Aid for Scientific Research (No. 61540133).

In [**1**, p. 58], Rudin posed the following problem; whether the above theorem is true for bounded real-analytic functions in $B$. In this paper, we produce a bounded real-analytic function in $B$ which gives a counterexample for the above problem.

THEOREM. *There exists a bounded real-analytic function $F$ in $B$ such that*
(i) *for each $z \in B$, there is a radius $r(z)$ $(0 < r(z) < 1)$ such that*

$$F(z) = \int_S F(\psi_z(r(z)\varsigma))\, d\sigma(\varsigma),$$

(ii) *$F$ is not $\mathscr{M}$-harmonic in $B$.*

To construct $F$, first we produce a bounded holomorphic function in the open unit disk $D$ in $C$ having a certain property (Lemma 2). We denote by $A(D)$ the disk algebra, the set of continuous functions in $\overline{D}$, the closed unit disk in $C$, which are holomorphic in $D$. For a bounded holomorphic function $f$ in $D$ (or $B$), let $\|f\|_\infty$ denote the supremum of $|f|$ in $D$. We shall say that a sequence of disjoint closed intervals $\{I_n\}_{n=1}^\infty$ in $[0,1)$ *converges to* 1, put $I_n = [r_n, R_n]$, if $0 \le r_1 \le R_1 < r_2 \le R_2 < \cdots$ and $r_n \to 1$ $(n \to \infty)$. For $0 < R < 1$, we put $\{|z| \le R\} = \{z \in C;\ |z| \le R\}$.

LEMMA 1. *Let $R$, $\delta$, $\sigma$ and $\eta$ be positive constants with $0 < R, \delta, \sigma, \eta < 1$. Let $f$ be a function in $A(D)$ satisfying the following conditions.*
(i) $\|f\|_\infty < 1$,
(ii) *$f > 0$ on $(-1,1)$ and $f(1) = 0$.*
*Let $\{I_n\}_{n=1}^\infty$ be a sequence of disjoint closed intervals in $[0,1)$ which converges to 1. Then there exist $g \in A(D)$, positive integers $p$ and $q$ $(p < q)$ such that*
(a) $\|f + g\|_\infty < 1$,
(b) *$g > 0$ on $(-1,1)$, hence $f + g > 0$ on $(-1,1)$, and $g(1) = 0$,*
(c) *$|g| < \delta$ on $\{|z| \le R\}$,*
(d) *$f + g < \sigma$ on $I_p$, and*
(e) *$g > 1 - \eta$ on $I_q$.*

PROOF. Let $\varepsilon$ be a positive constant with

$$0 < 3\varepsilon < \min\{\sigma, \eta\}. \tag{1}$$

Let $U$ be an open subset of $C$ such that $1 \in U$ and

$$|f| < \varepsilon \quad \text{on } \overline{D} \cap U. \tag{2}$$

By (ii), we may first choose a sufficient large positive integer $p$ such that

$$f < \sigma/2 \quad \text{on } I_p. \tag{3}$$

Put

$$h(z) = 1/(2 - z) \quad \text{for } z \in \overline{D}.$$

Then $h \in A(D)$, $h(1) = 1$, and

$$h > 0 \quad \text{on } (-1,1). \tag{4}$$

Since $|h(z)| < 1$ for $z \in \overline{D}$ with $z \ne 1$ and $\|f\|_\infty < 1$, there exists a positive integer $m$ such that

$$h^m(z) < \sigma/2 \quad \text{on } I_p, \tag{5}$$

$$|f| + |h^m| < 1 \quad \text{on } \overline{D}\backslash U, \quad \text{and} \tag{6}$$

$$|h^m| < \delta \quad \text{on } \{|z| \le R\}. \tag{7}$$

Since $h^m(z) \to 1$ $(z \to 1,\ z \in D)$, there exists a positive integer $q$ with $p < q$ such that

$$h^m > 1 - \varepsilon \quad \text{on } I_q. \tag{8}$$

For a real number $\lambda$ with $0 < \lambda < 1$, put

$$b_\lambda(z) = (z - \lambda)/(1 - \lambda z) \quad \text{for } z \in \overline{D}.$$

Then for each compact subset $K$ of $\overline{D}\backslash\{1\}$, $b_\lambda$ converges to $-1$ uniformly on $K$ with $\lambda \to 1$. Put

$$F_\lambda(z) = (1 - b_\lambda(z))/2 \quad \text{for } z \in \overline{D}.$$

Then

$$F_\lambda \in A(D) \quad \text{and} \quad F_\lambda(1) = 0, \tag{9}$$

$$\|F_\lambda\|_\infty = 1, \ \text{and} \tag{10}$$

$$F_\lambda > 0 \quad \text{on } (-1, 1). \tag{11}$$

Since $F_\lambda \to 1$ uniformly on $K$ $(\lambda \to 1)$, we may take $\lambda$, $0 < \lambda < 1$, such that

$$F_\lambda > 1 - \varepsilon \quad \text{on } I_q. \tag{12}$$

Put

$$g = (1 - \varepsilon) h^m F_\lambda. \tag{13}$$

We shall show that $g$ satisfies (a)–(e). By the definition of $g$, $g \in A(D)$ is clear. On $\overline{D}\backslash U$, we have

$$\begin{aligned} |f + g| &\le |f| + (1 - \varepsilon)|h^m|\,|F_\lambda| \quad \text{by (13)} \\ &< |f| + |h^m| < 1 \quad \text{by (10) and (6)}. \end{aligned}$$

On $\overline{D} \cap U$, we have

$$\begin{aligned} |f + g| &\le |f| + (1 - \varepsilon)|h^m|\,|F_\lambda| \\ &< \varepsilon + (1 - \varepsilon) \quad \text{by (2) and (10)} \\ &= 1. \end{aligned}$$

Thus $g$ satisfies (a).

By (4) and (11), the first part of (b) is satisfied. By (ii) and (9), the second part of (b) is satisfied. By (7) and (10), (c) is satisfied. By (3), (5), (10) and (11), (e) is satisfied.

On $I_q$, we have

$$\begin{aligned} g &= (1 - \varepsilon) h^m F_\lambda \\ &> (1 - \varepsilon)^3 \quad \text{by (8) and (12)} \\ &> 1 - 3\varepsilon \\ &> 1 - \eta \quad \text{by (1)}. \end{aligned}$$

Thus $g$ satisfies (e).

LEMMA 2. *Let $\{I_n\}_{n=1}^{\infty}$ be a sequence of disjoint closed intervals in $[0,1)$ which converges to 1. Then there is a bounded holomorphic function $G$ in $D$, and there is a subsequence $\{J_j\}_{j=1}^{\infty}$ of $\{I_n\}$ such that*

(a) $\|G\|_\infty \le 1$,

(b) $G > 0$ *on* $(-1,1)$,

(c, j) $\sup\{G(x); x \in J_{2j}\} < \inf\{G(x); x \in R, |x| \le R_{2j-1}\}$ *for* $j = 1,2,\dots$, *where $R_{2j-1}$ is the right end point of $J_{2j-1} = [r_{2j-1}, R_{2j-1}]$, and*

(d, j) $\inf\{G(x); x \in J_{2j+1}\} > \sup\{G(x); x \in R, |x| \le R_{2j-1}\}$ *for* $j = 1,2,\dots$.

PROOF. For a function $f$ in $D$ and $0 < R < 1$, we use the following notations.

$$\|f\|_R^{\sup} = \sup\{|f(z)|; |z| \le R\},$$
$$\|f\|_R^{\inf} = \inf\{|f(z)|; |z| \le R\}.$$

Put $J_1 = I_1 = [r_1, R_1]$. Let $f(z) = (1-z)/3$ for $z \in \overline{D}$. Then $f$ satisfies the assumptions of Lemma 1. Applying Lemma 1 for $R = R_1$,

$$\delta = \tfrac{1}{3}\min\{\|f\|_{R_1}^{\inf}, \tfrac{1}{2}(1 - \|f\|_{R_1}^{\sup})\},$$

$\sigma = \frac{1}{3}\|f\|_{R_1}^{\inf}$, and $\eta = \frac{1}{2}(1 - \|f\|_{R_1}^{\sup})$, there are $g_1 \in A(D)$, $J_2, J_3 \in \{I_n\}$ such that

$$\|f + g_1\|_\infty < 1, \tag{1}$$

$$g_1 > 0 \text{ on } (-1,1); \text{ hence } f + g_1 > 0 \text{ on } (-1,1), \tag{2}$$

$$(f + g_1)(z) \to 0 \qquad (z \to 1), \tag{3}$$

$$|g_1| < \tfrac{1}{3}\min\{\|f\|_{R_1}^{\inf}, \ \tfrac{1}{2}(1 - \|f\|_{R_1}^{\sup})\} \quad \text{on } \{|z| \le R_1\}, \tag{4}$$

$$f + g_1 < \tfrac{1}{3}\|f\|_{R_1}^{\inf} \quad \text{on } J_2, \tag{5}$$

$$g_1 > \tfrac{1}{2}(1 + \|f\|_{R_1}^{\sup}) \quad \text{on } J_3. \tag{6}$$

By (1), (2) and (3), we may apply Lemma 1 again for $R = R_3$, where $J_3 = [r_3, R_3]$. Hence there are $g_2 \in A(D)$, $J_4, J_5 \in \{I_n\}$ such that

$$\|f + g_1 + g_2\|_\infty < 1, \tag{7}$$
$$g_2 > 0 \text{ on } (-1,1); \text{ hence } f + g_1 + g_2 > 0 \text{ on } (-1,1), \tag{8}$$
$$(f + g_1 + g_2)(z) \to 0 \qquad (z \to 1), \tag{9}$$

$$|g_2| < \tfrac{1}{3}\min\left\{(\tfrac{1}{3})^{2-i}\|f\|_{R_{2i-1}}^{\inf}, (\tfrac{1}{2})^{3-i}\left(1 - \left\|f + \sum_{n=0}^{i-1} g_n\right\|_{R_{2i-1}}^{\sup}\right); i = 1,2\right\} \tag{10}$$
$$\text{on } \{|z| \le R_3\}, \text{ where } g_0 = 0,$$

$$f + g_1 + g_2 < \tfrac{1}{3}\|f\|_{R_3}^{\inf} \quad \text{on } J_4, \tag{11}$$
$$g_2 > \tfrac{1}{2}(1 + \|f + g_1\|_{R_3}^{\sup}) \quad \text{on } J_5. \tag{12}$$

We can continue this process successively. As a consequence, for each positive integer $j$, there are $g_j \in A(D)$, $J_{2j}, J_{2j+1} \in \{I_n\}$ such that

$$\left\| f + \sum_{n=0}^{j} g_n \right\|_\infty < 1, \tag{13}$$

$$g_j > 0 \text{ on } (-1,1); \text{ hence } f + \sum_{n=0}^{j} g_n > 0 \text{ on } (-1,1), \tag{14}$$

$$\left( f + \sum_{n=0}^{j} g_n \right)(z) \to 0 \qquad (z \to 1), \tag{15}$$

(16)

$$|g_j| < \tfrac{1}{3} \min \left\{ (\tfrac{1}{3})^{j-i} \|f\|_{R_{2i-1}}^{\inf}, (\tfrac{1}{2})^{j+1-i} \left( 1 - \left\| f + \sum_{n=0}^{i-1} g_n \right\|_{R_{2i-1}}^{\sup} \right) ; \ i = 1, 2, \ldots, j \right\}$$
$$\text{on } \{|z| \le R_{2j-1}\},$$

$$f + \sum_{n=0}^{j} g_n < \tfrac{1}{3} \|f\|_{R_{2j-1}}^{\inf} \quad \text{on } J_{2j}, \tag{17}$$

$$g_j > \frac{1}{2} \left( 1 + \left\| f + \sum_{n=0}^{j-1} g_n \right\|_{R_{2j-1}}^{\sup} \right) \quad \text{on } J_{2j+1}. \tag{18}$$

By (16), $|g_j| < (\frac{1}{3})^j \|f\|_{R_1}^{\inf}$ on $\{|z| \le R_{2j-1}\}$. Since $R_{2j-1} \to 1$ $(j \to \infty)$, $\sum_{n=0}^{\infty} g_n$ converges absolutely on every compact subset of $D$. Thus we may put

$$G(z) = f(z) + \sum_{n=0}^{\infty} g_n(z) \quad \text{for } z \in D.$$

Then $G$ is a holomorphic function in $D$. By (13), $G$ satisfies (a). By (14), $G$ satisfies (b).

We have

$$\begin{aligned}
&\sup\{G(x); x \in J_{2j}\} \\
&\quad \le \sup\left\{ \left( f + \sum_{n=0}^{j} g_n \right)(x); x \in J_{2j} \right\} + \sup\left\{ \left( \sum_{n=j+1}^{\infty} g_n \right)(x); x \in J_{2j} \right\} \\
&\quad \le \tfrac{1}{3} \|f\|_{R_{2j-1}}^{\inf} + \sum_{n=j+1}^{\infty} \|g_n\|_{R_{2n-1}}^{\sup} \quad \text{by (17)} \\
&\qquad\qquad (\text{since } J_{2j} \subset \{|z| \le R_k\} \text{ for } k \ge 2(j+1) - 1) \\
&\quad \le \tfrac{1}{3} \|f\|_{R_{2j-1}}^{\inf} + \sum_{n=j+1}^{\infty} (\tfrac{1}{3})^{n+1-j} \|f\|_{R_{2j-1}}^{\inf} \quad \text{by (16)} \\
&\quad \le \|f\|_{R_{2j-1}}^{\inf} \sum_{k=1}^{\infty} (\tfrac{1}{3})^k = \tfrac{1}{2} \|f\|_{R_{2j-1}}^{\inf}.
\end{aligned}$$

While we have

$$\begin{aligned}
\inf\{G(x); x \in R, |x| \le R_{2j-1}\} &= \inf\left\{\left(f + \sum_{n=0}^{\infty} g_n\right)(x); x \in R, |x| \le R_{2j-1}\right\} \\
&\ge \inf\{f(x); x \in R, |x| \le R_{2j-1}\} \quad \text{by (14)} \\
&= \inf\{|f(z)|; |z| \le R_{2j-1}\} \\
&= \|f\|^{\inf}_{R_{2j-1}}.
\end{aligned}$$

Hence $G$ satisfies $(c, j)$.

We get

$$\begin{aligned}
\inf\{G(x); x \in J_{2j+1}\} &= \inf\left\{\left(f + \sum_{n=0}^{\infty} g_n\right)(x); x \in J_{2j+1}\right\} \\
&\ge \inf\{g_j(x); x \in J_{2j+1}\} \quad \text{by (14) and } f > 0 \text{ on } (-1,1) \\
&> \frac{1}{2}\left(1 + \left\| f + \sum_{n=0}^{j-1} g_n \right\|^{\sup}_{R_{2j-1}}\right) \quad \text{by (18).}
\end{aligned}$$

While we get

$$\begin{aligned}
&\sup\{G(x); x \in R, |x| \le R_{2j-1}\} \\
&\quad = \sup\left\{\left(f + \sum_{n=0}^{\infty} g_n\right)(x); x \in R, |x| \le R_{2j-1}\right\} \\
&\quad \le \left\| f + \sum_{n=0}^{j-1} g_n \right\|^{\sup}_{R_{2j-1}} + \|g_j\|^{\sup}_{R_{2j-1}} + \sum_{n=j+1}^{\infty} \|g_n\|^{\sup}_{R_{2j-1}} \\
&\quad \le \left\| f + \sum_{n=0}^{j-1} g_n \right\|^{\sup}_{R_{2j-1}} + \frac{1}{6}\left(1 - \left\| f + \sum_{n=0}^{j-1} g_n \right\|^{\sup}_{R_{2j-1}}\right) \\
&\qquad + \frac{1}{3}\sum_{n=j+1}^{\infty}\left(\frac{1}{2}\right)^{n+1-j}\left(1 - \left\| f + \sum_{n=0}^{j-1} g_n \right\|^{\sup}_{R_{2j-1}}\right) \\
&\qquad\qquad (\text{by (16) and } R_{2j-1} \subset R_k \text{ for } k \ge 2(j+1) - 1) \\
&\quad \le \left\| f + \sum_{n=0}^{j-1} g_n \right\|^{\sup}_{R_{2j-1}} + \frac{1}{3}\left(1 - \left\| f + \sum_{n=0}^{j-1} g_n \right\|^{\sup}_{R_{2j-1}}\right) \\
&\quad = \frac{1}{3}\left(1 + 2\left\| f + \sum_{n=0}^{j-1} g_n \right\|^{\sup}_{R_{2j-1}}\right).
\end{aligned}$$

Since $\|f + \sum_{n=0}^{j-1} g_n\|^{\sup}_{R_{2j-1}} < 1$, we have

$$\frac{1}{3}\left(1 + 2\left\| f + \sum_{n=0}^{j-1} g_n \right\|^{\sup}_{R_{2j-1}}\right) < \frac{1}{2}\left(1 + \left\| f + \sum_{n=0}^{j-1} g_n \right\|^{\sup}_{R_{2j-1}}\right),$$

hence $G$ satisfies (d, j). This completes the proof.

PROOF OF THEOREM. First, we shall construct a sequence of disjoint closed intervals $\{I_n\}_{n=1}^{\infty}$ in $[0,1)$ which converges to 1 and satisfying specific conditions by induction.

For $a \in B$ and $0 < r < 1$, $\psi_a(rB)$ is an ellipsoid in $B$ and

$$\psi_a(rB) = \left\{ z \in B; \frac{|P_a z - c|^2}{r^2\rho^2} + \frac{|Q_a z|^2}{r^2\rho} < 1 \right\}$$

where

$$c = \frac{(1-r^2)a}{1-r^2|a|^2} \quad \text{and} \quad \rho = \frac{1-|a|^2}{1-r^2|a|^2}$$

(see [**1**, pp. 29–30]). Since $\psi_a$ is an automorphism $B = \bigcup\{\psi_a(rB); 0 < r < 1\}$ and $\psi_a(r_1 B) \subsetneqq \psi_a(r_2 B)$ if $0 < r_1 < r_2 < 1$. For $0 < R < 1$, put

$$\{|z| \le R\} = \left\{ z \in C^N; |z| = \left( \sum_{k=1}^{N} |z_k|^2 \right)^{1/2} \le R \right\}.$$

Here we use the following notation:

$$Q(R) = \sup_{|z| \le R} \inf\{\xi; 0 < \xi < 1, \psi_z(\xi B) \supset \{|\varsigma| \le R^{1/2}\}\}.$$

By the definition of $\psi_z$, $Q(R) < 1$ and $Q(R) \to 1$ $(R \to 1)$. Let $\xi_0$ be with $Q(R) < \xi_0 < 1$. Since $\psi_z(\xi_0 \eta)$ is a continuous function in $(z, \eta) \in B \times S$, $\bigcup\{\psi_z(\xi_0 S); |z| \le R\}$ is a connected set, hence $\{|w|^2; w \in \bigcup\{\psi_z(\xi_0 S); |z| \le R\}\}$ is a compact subinterval of $[0,1)$. By the definition of $Q(R)$, the above interval is disjoint with the interval $[0, R)$. If we take $\xi_0$ sufficiently close to 1, then the above interval is close to 1.

Using these properties, we shall choose $\{I_n\}$. We take $I_1 = [r_1', R_1']$ arbitrary with $0 \le r_1' < R_1' < 1$. Take $\xi_1$ as $Q(R_1') < \xi_1 < 1$. Put

$$I_2 = \left\{ |w|^2; w \in \bigcup\{\psi_z(\xi_1 S); |z| \le R_1'\} \right\}.$$

Then $I_2$ is a compact interval with $I_2 \subset [0,1)$ and $I_1 \cap I_2 = \varnothing$. Put $I_2 = [r_2', R_2']$ and take $\xi_2$ as $Q(R_2') < \xi_2 < 1$. Put

$$I_3 = \left\{ |w|^2; w \in \bigcup\{\psi_z(\xi_2 S); |z| \le R_2'\} \right\}.$$

Continuing this process, we can take a sequence of disjoint intervals $\{I_n\}$ such that

$$I_n = \left\{ |w|^2; w \in \bigcup\{\psi_z(\xi_{n-1} S); |z| \le R_{n-1}'\} \right\},$$

where $I_{n-1} = [r_{n-1}', R_{n-1}']$ and $Q(R_{n-1}') < \xi_{n-1} < 1$. Since we may take $\xi_n$ as $\xi_n \to 1$, $\{I_n\}$ converges to 1. This completes the construction of $\{I_n\}$.

By Lemma 2, there exist a bounded holomorphic function $G$ on $D$ and a subsequence $\{J_j\}_{j=1}^{\infty}$ of $\{I_n\}$ satisfying the assertion of Lemma 2. We put $J_j = [r_j, R_j]$. By our construction, for each positive integer $j$, there exist positive constants $\delta_j$ and $\sigma_j$ with $0 < \delta_j, \sigma_j < 1$ such that

$$J_{2j} \supset \left\{ |w|^2; w \in \bigcup\{\psi_z(\delta_j S); |z| \le R_{2j-1}\} \right\}, \tag{1}$$

$$J_{2j+1} \supset \left\{ |w|^2; w \in \bigcup\{\psi_z(\sigma_j S); |z| \le R_{2j-1}\} \right\}. \tag{2}$$

Put

$$F(z) = G(|z|^2), \quad \text{for } z \in B.$$

We shall show that $F$ is a desired function. By (b) of Lemma 2, $G_{|(-1,1)}$ is a positive real-analytic function on $(-1,1)$. Since $1 > |z|^2 = \sum_{k=1}^N x_k^2 + y_k^2$, where $z = (z_1, z_2, \ldots, z_N)$ and $z_k = x_k + iy_k$, $F(z)$ is a bounded real-analytic function in $B$.

To show (i), let $z \in B$. Then $|z| \le R_{2j-1}$ for some $j$. Then we have

$$\begin{aligned}
\int_S F(\psi_z(\delta_j \varsigma))\, d\sigma(\varsigma) &= \int_S G(|\psi_z(\delta_j \varsigma)|^2)\, d\sigma(\varsigma) \\
&\le \sup\{G(x); x \in J_{2j}\} \quad \text{by (1)} \\
&< \inf\{G(x); x \in R, |x| \le R_{2j-1}\} \quad \text{by (c, j) of Lemma 2} \\
&\le G(|z|^2) \quad (\text{since } |z|^2 \le |z| \le R_{2j-1}) \\
&= F(z).
\end{aligned}$$

On the other hand, we get

$$\begin{aligned}
\int_S F(\psi_z(\sigma_j \varsigma))\, d\sigma(\varsigma) &= \int_S G(|\psi_z(\sigma_j \varsigma)|^2)\, d\sigma(\varsigma) \\
&\ge \inf\{G(x); x \in J_{2j+1}\} \quad \text{by (2)} \\
&> \sup\{G(x); x \in R, |x| \le R_{2j-1}\} \quad \text{by (d, j) of Lemma 2} \\
&\ge G(|z|^2) \quad (\text{since } |z|^2 \le R_{2j-1}) \\
&= F(z).
\end{aligned}$$

Thus we get

$$\int_S F(\psi_z(\delta_j \varsigma))\, d\sigma(\varsigma) \le F(z) \le \int_S F(\psi_z(\sigma_j \varsigma))\, d\sigma(\varsigma).$$

Since $\int_S F(\psi_z(r\varsigma))\, d\sigma(\varsigma)$ is a continuous function with a variable $r$, $0 < r < 1$, there exists $r(z)$, $0 < r(z) < 1$, such that

$$F(z) = \int_S F(\psi_z(r(z)\varsigma))\, d\sigma(\varsigma).$$

Thus $F$ satisfies (i) of Theorem. Since

$$\int_S F(\psi_z(\sigma_j \varsigma))\, d\sigma(\varsigma) > F(z)$$

by the above estimates, $F$ does not have the invariant mean value property. Hence $F$ is not $\mathscr{M}$-harmonic. This completes the proof.

## References

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