PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 103, Number 3, July 1988

ASYMPTOTIC BEHAVIOR OF A CLASS OF NONLINEAR DIFFERENTIAL EQUATIONS OF nTH ORDER

QINGKAI KONG

(Communicated by Kenneth R. Meyer)

ABSTRACT. In this paper we obtain a result of the asymptotic behavior of the nth order equation $u^{(n)} + f(t, u, u', \ldots, u^{(n-1)}) = 0$ under some assumptions. For $n = 2$ and $f(t, u, u') \equiv f(t, u)$, it revises the result given by Jingcheng Tong, which is not true in general.

Much work has been done on the asymptotic behavior of the second order equation

$$u'' + f(t, u) = 0. \tag{1}$$

Some results of it are based on the integral inequalities of Gronwall-Bihari type. Here we quote the Theorem B in [**1**] as a proposition, according to the author, which includes the Theorem in [**2**] as its special case.

PROPOSITION. *Let $f(t, u)$ be continuous on D: $t \geq 0$, $-\infty < u < \infty$. If there are two nonnegative and continuous functions $v(t)$, $\phi(t)$ for $t \geq 0$, and a continuous function $g(u)$ for $u \geq 0$, such that*

(i) $\int_1^\infty v(t)\phi(t)\,dt < \infty$,

(ii) *for* $u > 0$, $g(u)$ *is positive and nondecreasing,*

(iii) $|f(t, u)| < v(t)\phi(t)g(|u|/t)$ *for* $t \geq 1$, $-\infty < u < \infty$,

then the equation (1) *has solutions which are asymptotic to $a + bt$, where a, b are constants and $b \neq 0$.*

According to the proof it seems to be true that every solution $u(t)$ of Equation (1) satisfies that $u'(t) \to b \in R$ as $t \to \infty$. But unfortunately, the Proposition does not hold in general because of two mistakes in its proof.

The first error arises because $G(c_3) + \int_1^t v(s)\phi(s)\,ds$ may be outside the domain of G^{-1} (see (4) in [**1**]). Subsequently, the second error arises because one cannot hence conclude that $\int_1^\infty |f(s, u(s))|\,ds < \infty$. For example, the equation $u'' - (2/t^4)u^2 = 0$ has a solution $u = t^2$ which does not satisfy the property $u'(t) \to b$ $(t \to \infty)$ as depicted in the proof of the Proposition. For the same reason, the example in [**1**] cannot be true.

In this paper we obtain a result of the asymptotic behavior of the nth order equation

$$u^{(n)} + f(t, u, u', \ldots, u^{(n-1)}) = 0. \tag{2}$$

In a special case, for $n = 2$ and $f(t, u, u') \equiv f(t, u)$, it revises the result in [**1**].

Received by the editors April 28, 1987 and, in revised form, July 8, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 34C99.

Here we cite a definition of a function class $\mathscr{F}$ given in [**3**, p. 207].

DEFINITION. A function $g(u)$ is said to belong to $\mathscr{F}$ if $g(u) > 0$ is nondecreasing and continuous on $(-\infty, \infty)$, and

$$g(u)/v \leq g(u/v) \qquad \text{for } u \geq 0,\ v \geq 1. \tag{3}$$

It is easy to see that $g(u) \in \mathscr{F}$ implies $\int_1^\infty (1/g(s))\, ds = \infty$. In fact, from (3), letting $v = u \geq 1$, we get $g(u)/u \leq g(1)$, i.e., $g(u) \leq g(1)u$, which follows

$$\int_1^\infty \frac{ds}{g(s)} \geq \int_1^\infty \frac{ds}{g(1)s} = \infty. \tag{4}$$

At the following we give an extension of the basic Bihari's inequality as a preliminary knowledge. It slightly modifies the Lemma in [**4**], which is not true for the case that $f(x) \geq 1$ is not satisfied.

LEMMA. *Assume that $f(t) > 0$ is nondecreasing and continuous on $[t_0, \infty)$, $h(t) \geq 1$ and $\phi(t) \geq 0$ are continuous on $[t_0, \infty)$, and $g(u) \in \mathscr{F}$. Let*

$$u(t) \leq f(t) + h(t) \int_{t_0}^t \phi(s) g(u(s))\, ds \tag{5}$$

for $t \geq t_0$. Then

$$u(t) \leq \frac{f(t)h(t)}{f(t_0)} G^{-1} \left(G(f(t_0)) + \int_{t_0}^t \phi(s)h(s)\, ds \right) \tag{6}$$

for $t \geq t_0$, where

$$G(u) = \int_{u_0}^u \frac{ds}{g(s)} \quad \text{for } u_0 > 0,\ u > 0.$$

PROOF. Without loss of generality, we assume that $u(t) \geq 0$. For otherwise, let $u_1 = \max\{u(x), 0\}$. Then $u_1(t)$ satisfies (5), and $u(t) \leq u_1(t)$. In view of that $f(t)$ is nondecreasing and $h(t) \geq 1$, from (5) we see

$$\frac{u(t)}{f(t)h(t)} \leq 1 + \int_{t_0}^t \phi(s) \frac{1}{f(s)} g(u(s))\, ds,$$

which follows

$$\begin{aligned} \frac{f(t_0)}{f(t)h(t)} u(t) &\leq f(t_0) + \int_{t_0}^t \phi(s)h(s) \frac{f(t_0)}{f(s)h(s)} g(u(s))\, ds \\ &\leq f(t_0) + \int_{t_0}^t \phi(s)h(s) g\left(\frac{f(t_0)}{f(s)h(s)} u(s) \right) ds. \end{aligned}$$

By using Bihari's inequality we get

$$\frac{f(t_0)}{f(t)h(t)} u(t) \leq G^{-1} \left(G(f(t_0)) + \int_{t_0}^t \phi(s)h(s)\, ds \right).$$

Then (6) is true. Noting $\int_1^\infty (1/g(s))\, ds = \infty$, we know that

$$G(f(t_0)) + \int_{t_0}^t \phi(s)h(s)\, ds \in \operatorname{Dom}(G^{-1}) \quad \text{for } t \geq t_0.$$

THEOREM. *Assume* $f(t, u_0, u_1, \ldots, u_{n-1})$ *is continuous on* $[1, \infty) \times (-\infty, \infty)^n$, *and*

$$|f(t, u_0, u_1, \ldots, u_{n-1})| \leq \sum_{i=0}^{n-1} \phi_i(t) g_i(|u_i|/t^{n-i-1}) \tag{7}$$

for $t \geq 1$ *and* $-\infty < u < \infty$, *where* $\phi_i(t) \geq 0$ $(i = 0, 1, \ldots, n-1)$ *is continuous on* $[1, \infty)$, *and* $\int_1^\infty \phi_i(t)\,dt < \infty$; $g_i(u) \in \mathscr{F}$. *Then every solution* $u(t)$ *of equation* (2) *satisfies* $u^{(n-i)}(t)/t^{i-1} \to a_i \in R$ $(i = 1, 2, \ldots, n)$ *as* $t \to \infty$, *where* $u^{(0)}(t) = u(t)$. *Furthermore, if* $f(t, u_0, u_1, \ldots, u_{n-1})$ *does not change its sign when* $u_i > 0$ $(i = 1, 2, \ldots, n-1)$ *and* $t \geq 1$, *then equation* (2) *has solutions such that* $a_i > 0$ $(i = 1, 2, \ldots, n)$.

PROOF. We denote that $G_i(u) = \int_1^u ds/g_i(s)$.

(i) Noting that $\int_1^\infty \phi_i(s)\,ds < \infty$, according to the Lemma, we can prove by induction that every solution $u(t)$ of equation (2) satisfies

$$\frac{|u^{(n-r)}(t)|}{t^{r-1}} \leq c_{n-r} + d_{n-r} \int_1^t \sum_{i=0}^{n-r-1} \phi_i(s) g_i\left(\frac{|u^{(i)}(s)|}{s^{n-i-1}}\right) ds, \qquad r = 1, 2, \ldots, n-1 \tag{8}$$

for $t \geq 1$, where $c_{n-r} > 0$, $d_{n-r} > 0$.

In fact, from (2) we have

$$u^{(n-1)}(t) = c - \int_1^t f(s, u(s), u'(s), \ldots, u^{(n-1)}(s))\,ds.$$

Choose c' such that $c' \geq |c|$ and $c' > 0$. From (7) we get

$$\begin{aligned} |u^{(n-1)}(t)| \leq {} & \left[c' + \int_1^t \sum_{i=0}^{n-2} \phi_i(s) g_i\left(\frac{|u^{(i)}(s)|}{s^{n-i-1}}\right) ds\right] \\ & + \int_1^t \phi_{n-1}(s) g_{n-1}(|u^{(n-1)}(s)|)\,ds. \end{aligned}$$

Using the Lemma we get

$$\begin{aligned} |u^{(n-1)}(t)| \leq {} & \left[1 + \frac{1}{c'} \int_1^t \sum_{i=0}^{n-2} \phi_i(s) g_i\left(\frac{|u^{(i)}(s)|}{s^{n-i-1}}\right) ds\right] \\ & \times G_{n-1}^{-1}\left(G_{n-1}(c') + \int_{t_0}^t \phi_{n-1}(s)\,ds\right) \\ \leq {} & c_{n-1} + d_{n-1} \int_1^t \sum_{i=0}^{n-2} \phi_i(s) g_i\left(\frac{|u^{(i)}(s)|}{s^{n-i-1}}\right) ds, \end{aligned}$$

where

$$c_{n-1} = G_{n-1}^{-1}\left(G_{n-1}(c') + \int_1^\infty \phi_{n-1}(s)\,ds\right) > 0, \qquad d_{n-1} = c_{n-1}/c' > 0.$$

Thus (8) is true for $r = 1$. Suppose (8) is true for $r = k < n-1$, i.e.,

$$\frac{|u^{(n-k)}(t)|}{t^{k-1}} \leq c_{n-k} + d_{n-k} \int_1^t \sum_{i=0}^{n-k-1} \phi_i(s) g_i\left(\frac{|u^{(i)}(s)|}{s^{n-i-1}}\right) ds,$$

where $c_{n-k} > 0$, $d_{n-k} > 0$. Then

$$|u^{(n-k)}(t)| \le c_{n-k}t^{k-1} + d_{n-k}t^{k-1}\int_1^t \sum_{i=0}^{n-k-1} \phi_i(s) g_i\left(\frac{|u^{(i)}(s)|}{s^{n-i-1}}\right) ds.$$

Considering that

$$|u^{(n-k-1)}(t)| - |u^{(n-k-1)}(1)| \le \int_1^t |u^{(n-k)}(s)|\, ds,$$

we have

$$|u^{(n-k-1)}(t)| \le c'_{n-k}t^k + c''_{n-k} + d'_{n-k}\int_1^t (t^k - s^k) \sum_{i=0}^{n-k-1} \phi_i(s) g_i\left(\frac{|u^{(i)}(s)|}{s^{n-i-1}}\right) ds,$$

where

$$c'_{n-k} = \frac{c_{n-k}}{k}, \quad c''_{n-k} = |u^{(n-k-1)}(1)| - \frac{c_{n-k}}{k}, \quad d'_{n-k} = \frac{d_{n-k}}{k}.$$

Let $c'''_{n-k} = c'_{n-k} + |c''_{n-k}|$, then

$$\frac{|u^{(n-k-1)}(t)|}{t^k} \le \left[c'''_{n-k} + d'_{n-k}\int_1^t \sum_{i=0}^{n-k-2} \phi_i(s) g_i\left(\frac{|u^{(i)}(s)|}{s^{n-i-1}}\right) ds\right] + d'_{n-k}\int_1^t \phi_{n-k-1}(s) g_{n-k-1}\left(\frac{|u^{(n-k-1)}(s)|}{s^k}\right) ds$$

for $t \ge 1$. Using the Lemma we get

$$\begin{aligned}\frac{|u^{(n-k-1)}(t)|}{t^k} &\le \left[1 + \frac{d'_{n-k}}{c'''_{n-k}}\int_1^t \sum_{i=0}^{n-k-2} \phi_i(s) g_i\left(\frac{|u^{(i)}(s)|}{s^{n-i-1}}\right) ds\right] \\ &\quad \times G^{-1}_{n-k-1}\left(G_{n-k-1}(c'''_{n-k}) + d'_{n-k}\int_1^t \phi_{n-k-1}(s)\, ds\right) \\ &\le c_{n-k-1} + d_{n-k-1}\int_1^t \sum_{i=0}^{n-k-2} \phi_i(s) g_i\left(\frac{|u^{(i)}(s)|}{s^{n-i-1}}\right) ds,\end{aligned}$$

where $c_{n-k-1} = G^{-1}_{n-k-1}(G_{n-k-1}(c''') + d'_{n-k}\int_1^\infty \phi_{n-k-1}(s)\, ds) > 0$, $d_{n-k-1} = d'_{n-k}c_{n-k-1}/c'''_{n-k} > 0$. Thus (8) is true for $r = k+1$.

(ii) By induction we can also prove that

$$|u^{(i)}(t)|/t^{n-i-1} \le b_i, \qquad i = 0, 1, \dots, n-1, \tag{9}$$

for $t \ge 1$, where b_i ($i = 0, 1, \dots, n-1$) are constants. In fact, from (8), letting $r = n-1$, we get

$$\frac{|u'(t)|}{t^{n-2}} \le c_1 + d_1\int_1^t \phi_0(s) g_0\left(\frac{|u(s)|}{s^{n-1}}\right) ds,$$

which follows

$$|u'(t)| \le c_1 t^{n-2} + d_1 t^{n-2}\int_1^t \phi_0(s) g_0\left(\frac{|u(s)|}{s^{n-1}}\right) ds.$$

Considering that

$$|u(t)| - |u(1)| \le \int_1^t |u'(s)|\,ds,$$

we have

$$|u(t)| \le c_1' t^{n-1} + c_1'' + d_0 \int_1^t (t^{n-1} - s^{n-1})\phi_0(s) g_0\left(\frac{|u(s)|}{s^{n-1}}\right) ds,$$

where $c_1' = c_1/(n-1)$, $c_1'' = |u(1)| - c_1/(n-1)$, $d_0 = d_1/(n-1)$. Let $c_0 = c_1' + |c_1''|$. Then

$$\frac{|u(t)|}{t^{n-1}} \le c_0 + d_0 \int_1^t \phi_0(s) g_0\left(\frac{|u(s)|}{s^{n-1}}\right) ds$$

for $t \ge 1$. Using the Lemma we get

$$\begin{aligned} |u(t)|/t^{n-1} &\le G_0^{-1}\left(G_0(c_0) + d_0 \int_1^t \phi_0(s)\,ds\right) \\ &\le G_0^{-1}\left(G_0(c_0) + d_0 \int_1^\infty \phi_0(s)\,ds\right) \triangleq b_0. \end{aligned}$$

Thus (9) is true for $i = 0$. Suppose (9) is true for $i = k < n-1$. From (8), letting $r = n-k-2$, we have

$$\begin{aligned} \frac{|u^{(k+2)}(t)|}{t^{n-k-3}} &\le c_{k+2} + d_{k+2} \int_1^t \sum_{i=0}^k \phi_i(s) g_i(b_i)\,ds \\ &\quad + d_{k+2} \int_1^t \phi_{k+1}(s) g_{k+1}\left(\frac{|u^{(k+1)}(s)|}{s^{n-k-2}}\right) ds \\ &\le c_{k+2}' + d_{k+2} \int_1^t \phi_{k+1}(s) g_{k+1}\left(\frac{|u^{(k+1)}(s)|}{s^{n-k-2}}\right) ds, \end{aligned}$$

where $c_{k+2}' = c_{k+2} + d_{k+2} \int_1^\infty \sum_{i=0}^k \phi_i(s) g_i(b_i)\,ds$. Hence

$$|u^{(k+2)}(t)| \le c_{k+2}' t^{n-k-3} + d_{k+2} t^{n-k-3} \int_1^t \phi_{k+1}(s) g_{k+1}\left(\frac{|u^{(k+1)}(s)|}{s^{n-k-2}}\right) ds.$$

Considering that

$$|u^{(k+1)}(t)| - |u^{(k+1)}(1)| \le \int_1^t |u^{(k+2)}(s)|\,ds,$$

we have

$$\begin{aligned} |u^{(k+1)}(t)| &\le c_{k+2}'' t^{n-k-2} \\ &\quad + c_{k+2}''' + d_{k+1} \int_1^t (t^{n-k-2} - s^{n-k-2})\phi_{k+1}(s) g_{k+1}\left(\frac{|u^{(k+1)}(s)|}{s^{n-k-2}}\right) ds, \end{aligned}$$

where $c_{k+2}'' = c_{k+2}'/(n-k-2)$, $c_{k+2}''' = |u^{(k+1)}(1)| - c_{k+2}'/(n-k-2)$, $d_{k+1} = d_{k+2}/(n-k-2)$. Letting $c_{k+1} = c_{k+2}'' + |c_{k+2}'''|$, then

$$\frac{|u^{(k+1)}(t)|}{t^{n-k-2}} \le c_{k+1} + d_{k+1} \int_1^t \phi_{k+1}(s) g_{k+1}\left(\frac{|u^{(k+1)}(s)|}{s^{n-k-2}}\right) ds$$

for $t \geq 1$. Using the Lemma we get

$$|u^{(k+1)}(t)|/t^{n-k-2} \leq G_{k+1}^{-1}\left(G_{k+1}(c_{k+1}) + d_{k+1}\int_1^t \phi_{k+1}(s)\,ds\right)$$
$$\leq G_{k+1}^{-1}\left(G_{k+1}(c_{k+1}) + d_{k+1}\int_1^\infty \phi_{k+1}(s)\,ds\right) \triangleq b_{k+1}.$$

Thus (9) is true for $i = k+1$.

(iii) From equation (2) we obtain that $i = 1, 2, \ldots, n$

(10)

$$u^{(n-i)}(t) = \sum_{j=1}^{i} \frac{d_{ji}}{(i-j)!}t^{i-j} - \frac{1}{(i-1)!}\int_1^t (t-s)^{i-1} f(s, u(s), u'(s), \ldots, u^{(n-1)}(s))\,ds,$$

where $d_{ji} = u^{(n-1)}(1)$. And from (9) we know that when $t \geq 1$,

$$\int_1^t \left(1 - \frac{s}{t}\right)^{i-1} |f(s, u(s), u'(s), \ldots, u^{(n-1)}(s))|\,ds$$
$$\leq \sum_{i=0}^{n-1} \int_1^t \phi_i(s) g_i\left(\frac{|u^{(i)}(s)|}{s^{n-i-1}}\right)\,ds$$
$$\leq \sum_{i=0}^{n-1} g_i(b_i)\int_1^\infty \phi_i(s)\,ds < \infty.$$

Since $\int_1^t (1-s/t)^{i-1}|f(s, u(s), u'(s), \ldots, u^{(n-1)}(s))|\,ds$ is nondecreasing in t, then

$$\lim_{t\to\infty} \int_1^t \left(1 - \frac{s}{t}\right)^{i-1} |f(s, u(s), u'(s), \ldots, u^{(n-1)}(s))|\,ds$$

exists, which follows that

$$\lim_{t\to\infty} \int_1^t \left(1 - \frac{s}{t}\right)^{i-1} f(s, u(s), u'(s), \ldots, u^{(n-1)}(s))\,ds = h_i$$

exists. According to L'Hospital's criterion it is easy to see that $h_1 = h_2 = \cdots = h_n \triangleq h$. Dividing both sides of (10) by t^{i-1} and letting $t \to \infty$, we get

$$\frac{u^{(n-i)}(t)}{t^{i-1}} \to \frac{d_{1i} - h_i}{(i-1)!} = \frac{u^{(n-1)}(1) - h}{(i-1)!} \triangleq a_i. \tag{11}$$

(iv) Let $f(t, u_0, \ldots, u_{n-1}) > 0$ for $u_i > 0$ $(i = 0, \ldots, n-1)$. For any $r > 0$, because $\int_1^\infty \phi_i(s)\,ds < \infty$, we can choose a T so large that

$$\sum_{i=0}^{n-1} g_i\left(\frac{r}{(n-i-1)!}\right)\int_T^\infty \phi_i(s)\,ds < \frac{r}{4}.$$

If $u(t)$ is the solution of equation (2) satisfying

$$u(T) = u'(T) = \cdots = u^{(n-2)}(T) = 0, \qquad u^{(n-1)}(T) = r, \tag{12}$$

then

(13)

$$u^{(n-i)}(t) = \frac{r}{(i-1)!}(t-T)^{i-1} - \frac{1}{(i-1)!}\int_T^t (t-s)^{i-1} f(s, u(s), \ldots, u^{(n-1)}(s))\,ds,$$

which follows for $t > T$

$$u^{(n-i)}(t)/t^{i-1} \leq r/(i-1)!, \qquad i = 1, 2, \ldots, n,$$

i.e.,

$$u^{(i)}(t)/t^{n-i-1} \leq r/(n-i-1)!, \qquad i = 0, 1, \ldots, n-1,$$

as long as $u^{(i)}(s) > 0$ $(i = 0, 1, \ldots, n-1)$ for $s \in (T, t)$. Thus there exists a maximal interval $(T, \tilde{T})$ in which

$$0 < u^{(i)}(t)/t^{n-i-1} \leq r/(n-i-1)!, \quad \text{and} \quad u^{(n-1)}(t) > r/2.$$

However it follows $\tilde{T} = \infty$. For otherwise

$$\begin{aligned} u^{(n-1)}(\tilde{T}) &= r - \int_T^{\tilde{T}} f(s, u(s), \ldots, u^{(n-1)}(s))\, ds \\ &\geq r - \sum_{i=0}^{n-1} \int_T^{\tilde{T}} \phi_i(s) g_i \left(\frac{|u^{(i)}(s)|}{s^{n-i-1}} \right) ds \\ &\geq r - \sum_{i=0}^{n-1} g_i \left(\frac{r}{(n-i-1)!} \right) \int_T^{\tilde{T}} \phi_i(s)\, ds \geq \frac{3}{4} r. \end{aligned}$$

It would contradict the maximal property of $\tilde{T}$. This means $u^{(n-1)}(t) > r/2$ for $T \leq t < \infty$. Hence $a_1 \geq r/2 > 0$. From (11) we know $a_i > 0$ $(i = 1, 2, \ldots, n)$.

Let $f(t, u_0, \ldots, u_{n-1}) < 0$ for $u_i > 0$ $(i = 0, \ldots, n-1)$. Then we can conclude that $u^{(n-i)}(t) > 0$ $(i = 1, 2, \ldots, n)$ for $t > T$. For otherwise there would be a $\tilde{T} > T$ such that $u^{(n-i)}(t) > 0$ for $t \in (T, \tilde{T})$ and $u^{(n-k)}(\tilde{T}) = 0$ for some k. In view of (13) we have

$$\begin{aligned} u^{(n-k)}(\tilde{T}) = {} & \frac{r}{(k-1)!} (\tilde{T} - T)^{k-1} \\ & - \frac{1}{(k-1)!} \int_T^{\tilde{T}} (\tilde{T} - S)^{k-1} f(s, u(s), \ldots, u^{(n-1)}(s))\, ds > 0, \end{aligned}$$

which contradicts the assumption. Thus from (13) we know

$$\frac{u^{(n-i)}(t)}{t^{i-1}} \geq \frac{r}{(i-1)!} 1 - \left(1 - \frac{T}{t}\right)^{i-1} \quad \text{for } t \geq T.$$

This shows $a_i \geq r/2(i-1)! > 0$.

The proof is complete.

EXAMPLE. Consider the equation

$$u'' - (2/t^3)(|uu'|)^{1/2} = 0. \tag{14}$$

We have $f(t, u, u') = -(2/t^3)(|uu'|)^{1/2}$. Hence

$$|f(t, u, u')| \leq (1/t^2)(|u|/t + |u'|)$$

for $t \geq 1$. Let $\phi_i(t) = 1/t^2$, $g_i(u) = u$ $(i = 0, 1)$. The conditions of the Theorem are satisfied. By using the theorem, we see, for every solution $u(t)$ of equation (14)

$$u'(t) \to a_1, \qquad u(t)/t \to a_2 = a_1 \quad \text{as } t \to \infty.$$

And equation (13) has solutions such that $a_1 > 0$.

The conclusion cannot be drawn from [**5**] and other literatures.

ACKNOWLEDGMENT. The author is grateful to the referee for correcting errors in the initial manuscript and helpful suggestions.

REFERENCES

1. Jingcheng Tong, *The asymptotic behavior of a class of nonlinear differential equations of second order*, Proc. Amer. Math. Soc. **84** (1982), 235–236.
2. D. S. Cohen, *The asymptotic behavior of a class of nonlinear differential equations*, Proc. Amer. Math. Soc. **18** (1967), 607–609.
3. P. R. Beesack, *On Lakshmikantham's comparison method for Gronwall inequalities*, Ann. Polon. Math. **35** (1977), 187–222.
4. U. D. Dhongade and S. G. deo, *A nonlinear generalization of Bihari's inequality*, Proc. Amer. Math. Soc. **54** (1976), 211–216.
5. K. Kreith, *Extremal solutions for a class of nonlinear differential equations*, Proc. Amer. Math. Soc. **79** (1980), 415–421.

DEPARTMENT OF APPLIED MATHEMATICS, QINGDAO UNIVERSITY OF OCEANOGRAPHY, QINGDAO, THE PEOPLE'S REPUBLIC OF CHINA