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# GENERATING FUNCTIONS FOR RELATIVES OF CLASSICAL POLYNOMIALS

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ABSTRACT. For several classical polynomials $u_n(x)$ satisfying a second order linear differential equation $D_n(x)$, there is a generating function $u(x,t) = \sum_{n=0}^{\infty} u_n(x)t^n$. We provide expansions $v(x,t) = \sum_{n=0}^{\infty} v_n(x)t^n$ where $v_n(x)$ is a second solution of $D_n(x)$.

**1. Introduction.** For the derivation of generating functions we refer to [**4** and **3**]; for lists of them we refer to [**2**], and for orthogonal polynomials in particular to [**1**] also. There are very well-known generating functions $u(x,t)$ for the Hermite polynomials $He_n(x)$ [**4**, p. 83], generalised Laguerre polynomials $L_n^{(\alpha)}(x)$ [**4**, p. 84], special generalised Laguerre polynomials $(-1)^n L_n^{(-\alpha-n)}(x)$ [**4**, p. 84] and the Gegenbauer or ultraspherical polynomials $C_n^{(\alpha)}(x)$ [**4**, p. 83]; there are also hypergeometric polynomials $g_n^{(\alpha,\beta)}(x,1)$ named after Lagrange in [**4**, p. 85], for which see also [**2**, (16), p. 247].

Each of these polynomials satisfies a well-known homogeneous linear differential equation of order 2, for which there will be second, linearly independent, solutions. The problem which we address is that of providing a generating function for a sequence of such second solutions.

We deal in detail with the Hermite polynomials and state corresponding results for the others.

**2. Preliminaries.** If

$$u(x,t) = \exp(xt - \tfrac{1}{2}t^2) \tag{2.1}$$

then

$$\frac{\partial u}{\partial x} = tu, \quad \frac{\partial^2 u}{\partial x^2} = t^2 u, \quad \frac{\partial u}{\partial t} = (x-t)u$$

and so $w = u(x,t)$ satisfies the equation

$$\frac{\partial^2 w}{\partial x^2} - x\frac{\partial w}{\partial x} + t\frac{\partial w}{\partial t} = 0. \tag{2.2}$$

Now $u(x,t)$ is an entire function in the variable $t$ so there is an expansion

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x)t^n \tag{2.3}$$

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where $u_n(x) = He_n(x)/n!$. On applying (2.2) to (2.3), differentiating termwise with respect to $x$ and $t$, we see that for all $t \in \mathbf{C}$

$$0 = \sum_{n=0}^{\infty} [u_n''(x) - xu_n'(x) + nu_n(x)]t^n,$$

and so for all $n \geq 0$, $w_n(x) = u_n(x)$ satisfies

$$w_n''(x) - xw_n'(x) + nw_n(x) = 0 \tag{2.4}$$

which is Hermite's differential equation [**1**, 22.6.21]. We seek an expansion

$$v(x,t) = \sum_{n=0}^{\infty} v_n(x)t^n \tag{2.5}$$

such that, for each fixed $n$, $u_n$ and $v_n$ are linearly independent solutions of (2.4).

**3. A guiding example.** To prompt an approach to this, we consider a similar situation to (2.3) and (2.4) where now

$$u(x,t) = \frac{1 - t\cos x}{1 - 2t\cos x + t^2}, \qquad v(x,t) = \frac{t\sin x}{1 - 2t\cos x + t^2} \tag{3.1}$$

and

$$u_n(x) = \cos nx, \qquad v_n(x) = \sin nx. \tag{3.2}$$

These expansions are easily found from the geometric progression

$$(1 - \varsigma)^{-1} = \sum_{n=0}^{\infty} \varsigma^n$$

on using the polar coordinates $\varsigma = te^{ix}$, where $t$ and $x$ are real, and then using analytic continuation. Then $u_n$ and $v_n$ are linearly independent solutions of

$$w_n''(x) + n^2 w_n(x) = 0$$

which corresponds to (2.4).

Our heuristic approach is to consider whether we know of any other situation in which $u(x,t)$ and $v(x,t)$ in (3.1), or $u_n$ and $v_n$ in (3.2), or both, are linked. As

$$\frac{d^2}{dt^2}(1 - t\cos x) = 0, \quad \frac{d^2}{dt^2}(t\sin x) = 0 \tag{3.3}$$

we see from (3.1) that both $u(x,t)$ and $v(x,t)$ satisfy the equation

$$\frac{d^2}{dt^2}[(1 - 2t\cos x + t^2)w] = 0$$

which expands to

$$(1 - 2t\cos x + t^2)w''(t) + 4(-\cos x + t)w'(t) + 2w(t) = 0.$$

If we now apply to this the standard method of seeking for a homogeneous linear differential equation a solution of the form

$$w(t) = \sum_{n=-\infty}^{\infty} k_n t^{n+\sigma} \tag{3.4}$$

where $k_0 \neq 0$ and $k_n = 0$ for all $n < 0$, we find that we need to satisfy the difference equation

$$(3.5)\quad (n+\sigma)(n+\sigma-1)k_n - 2\cos x \cdot (n+\sigma)(n+\sigma-1)k_{n-1} + (n+\sigma)(n+\sigma-1)k_{n-2} = 0, \qquad n \in \mathbf{Z}.$$

This is trivially satisfied for all $n < 0$, and on putting $n = 0$ we obtain the indicial equation $\sigma(\sigma - 1) = 0$, with roots 0 and 1. As $u(x,0) = 1$, $u$ is a solution which corresponds to $\sigma = 0$ and on inserting this in (3.5) we see that $k_n = u_n(x)$ in (3.2) satisfies

$$(3.6)\quad n(n-1)u_n(x) - 2\cos x \cdot n(n-1)u_{n-1}(x) + n(n-1)u_{n-2}(x) = 0 \qquad (n \geq 1),$$

with $u_{-1}(x) = 0$. Similarly as $v(x,0) = 0$, $v$ is a solution for which $\sigma = 1$ so on inserting this in (3.5) we see that $k_n = v_{n+1}(x)$ in (3.2) satisfies

$$(n+1)nv_{n+1}(x) - 2\cos x \cdot (n+1)nv_n(x) + (n+1)nv_{n-1}(x) = 0,$$

with $v_{-1}(x) = 0$, and so

$$(3.7)\quad n(n-1)v_n(x) - 2\cos x \cdot n(n-1)v_{n-1}(x) + n(n-1)v_{n-2}(x) = 0 \qquad (n \geq 2).$$

If we inspect (3.6) and (3.7) we see that it is the term $n$ in the coefficient of $u_n(x)$ in (3.6) that enables us to accommodate to the stipulation $k_0 \neq 0$ and so obtain one nontrivial solution of (3.5). Similarly it is the presence of the term $n-1$ in the coefficients of $v_n(x)$ and $v_{n-1}(x)$ in (3.7) that enables a second nontrivial solution to be found.

Now returning to our original problem concerning Hermite polynomials, we note that $w = u(x,t)$ in (2.1) satisfies the differential equation

$$tw'(t) + (-xt + t^2)w(t) = 0$$

and for (3.4) to furnish a solution of this we need

$$(3.8)\qquad (n+\sigma)k_n - xk_{n-1} + k_{n-2} = 0, \qquad n \in \mathbf{Z}.$$

If again we seek a solution with $k_0 \neq 0$ and $k_n = 0$ for all $n < 0$, we get the indicial equation $\sigma = 0$ and so

$$nk_n - xk_{n-1} + k_{n-2} = 0, \qquad n \geq 1,$$

holds for $k_n = u_n(x)$ in (2.3) for $n \geq 0$, with $u_{-1}(x) = 0$. As compared with (3.6) and (3.7) we have $n$ in the coefficient of $k_n$ for use when $n = 0$ to enable us to get one nontrivial solution, the one which we know, but we have not got a term $n-1$ in the coefficients of $k_n$ and $k_{n-1}$ which would allow a second solution. This suggests that we should work with the difference equation

$$(3.9)\qquad (n+\sigma-1)[(n+\sigma)k_n - xk_{n-1} + k_{n-2}] = 0$$

instead of (3.8).

Now working backwards, we see that the differential equation for which (3.4) will provide a solution if (3.9) holds, is

$$\left(t\frac{d}{dt} - 1\right)\left[t\frac{dw}{dt} + (-xt + t^2)w\right] = 0$$

the solutions to which have the form

$$(3.10)\qquad w(x,t) = e^{xt - t^2/2}\left[c_2 + c_1\int_0^t e^{-xs + s^2/2}\,ds\right],$$

where $c_1$ and $c_2$ are arbitrary constants.

**4. Solution for Hermite's equation.** With $w(x,t)$ as in (3.10) we now try to show that we again satisfy (2.2) and thus lead on to (2.4). However, instead we are led to a differential equation of order 4 (instead of (2.4)) when $c_1 \neq 0$, $c_2 = 0$. For a way out of this disappointment we look at (3.3) and note that although

$$\frac{d^2}{dt^2}(c_2 + c_1 t) = 0,$$

it is not arbitrary constants that occur there as in each case $c_1$ depends on $x$. Accordingly we now take

$$w(x,t) = e^{xt-t^2/2}\left[c_2(x) + c_1(x)\int_0^t e^{-xs+s^2/2}\,ds\right] \tag{4.1}$$

and try to choose $c_1(x)$ and $c_2(x)$ so that (2.2) is satisfied. If desired the previous work can be regarded as motivation, and the calculations carefully checked only from here on.

From (4.1)

$$\frac{\partial w}{\partial x} = tw + e^{xt-t^2/2}\left[c_2'(x) + c_1'(x)\int_0^t e^{-xs+s^2/2}\,ds - c_1(x)\int_0^t se^{-xs+s^2/2}\,ds\right]$$

and on integration by parts

$$\frac{\partial w}{\partial x} = tw + e^{xt-t^2/2}\left\{c_2'(x) + c_1(x) - c_1(x)e^{-xt+t^2/2} + [c_1'(x) - xc_1(x)]\int_0^t e^{-xs+s^2/2}\,ds\right\}. \tag{4.2}$$

We choose $c_1(x)$ to satisfy

$$c_1'(x) - xc_1(x) = 0$$

so that we can take

$$c_1(x) = e^{x^2/2}.$$

We next choose $c_2(x)$ so that

$$c_2'(x) + c_1(x) = 0$$

and so can take

$$c_2(x) = -\int_0^x e^{s^2/2}\,ds.$$

With this choice in (4.1) we get in (4.2) that

$$\frac{\partial w}{\partial x} = tw - c_1(x) = tw - e^{x^2/2}. \tag{4.3}$$

From this

$$\frac{\partial^2 w}{\partial x^2} = t\frac{\partial w}{\partial x} - xe^{x^2/2} = t^2 w - (t+x)e^{x^2/2},$$

so that

$$\frac{\partial^2 w}{\partial x^2} - x\frac{\partial w}{\partial x} = t^2 w - xtw - te^{x^2/2}.$$

But by (4.1),

$$\frac{\partial w}{\partial t} = (x-t)w + c_1(x)$$

so that

$$t\frac{\partial w}{\partial t} = xtw - t^2 w + te^{x^2/2} = -\frac{\partial^2 w}{\partial x^2} + x\frac{\partial w}{\partial x}$$

and we have (2.2) again. Accordingly in

$$\text{(4.4)} \qquad v(x,t) = e^{xt-t^2/2}\left[-\int_0^x e^{s^2/2}\,ds + e^{x^2/2}\int_0^t e^{-xs+s^2/2}\,ds\right] = \sum_{n=0}^{\infty} v_n(x)t^n,$$

the coefficients $v_n$ satisfy the Hermite differential equation (2.4).

**5. Linear independence.** A pair of linearly independent solutions of the equation (2.4) is $M(-\frac{1}{2}n, \frac{1}{2}; \frac{1}{2}x^2)$, $xM(\frac{1}{2} - \frac{1}{2}n, \frac{3}{2}; \frac{1}{2}x^2)$ where

$$M(a,b;z) = \sum_{m=0}^{\infty} \frac{(a)_m}{m!(b)_m} z^m$$

is the well-known Kummer function [**1**, 13.1.2]. Accordingly any solution $W(x)$ of (2.4), in a neighbourhood of the origin, can be expressed as a linear combination of these,

$$W(x) = c_1 M(-\tfrac{1}{2}n, \tfrac{1}{2}; \tfrac{1}{2}x^2) + c_2 x M(\tfrac{1}{2} - \tfrac{1}{2}n, \tfrac{3}{2}; \tfrac{1}{2}x^2)$$

for some constants $c_1$ and $c_2$, and in fact

$$\text{(5.1)} \qquad c_1 = W(0), \qquad c_2 = W'(0).$$

This can be applied to $u(x,t)$ in (2.1) to prove the results [**1**, 22.5.56-57] that for $m \geq 0$

$$\text{(5.2)} \qquad \begin{aligned} u_{2m}(x) &= \frac{(-1/2)^m}{m!} M(-m, \tfrac{1}{2}; \tfrac{1}{2}x^2), \\ u_{2m+1}(x) &= \frac{(-1/2)^m}{m!} x M(-m, \tfrac{1}{2}; \tfrac{1}{2}x^2). \end{aligned}$$

Now from (4.4)

$$\begin{aligned} v(0,t) &= e^{-t^2/2}\int_0^t e^{s^2/2}\,ds = te^{-t^2/2}\int_0^1 e^{t^2r^2/2}\,dr \\ &= t\sum_{m=0}^{\infty} \frac{(-t^2/2)^m}{m!}\left[\int_0^1 (1-r^2)^m\,dr\right]. \end{aligned}$$

The integral in this is equal to

$$\frac{1}{2}\int_0^1 y^{-1/2}(1-y)^m\,dy = \frac{1}{2}B\left(\frac{1}{2}, m+1\right) = \frac{m!}{(3/2)_m}.$$

Hence

$$v(0,t) = \sum_{m=0}^{\infty} \frac{(-1)^m}{2^m (3/2)_m} t^{2m+1}$$

so that

$$\text{(5.3)} \qquad v_n(0) = \begin{cases} \dfrac{(-1/2)^m}{(3/2)_m}, & \text{if } n = 2m+1, \\ 0, & \text{if } n = 2m. \end{cases}$$

Moreover from (4.3) and (4.4)

$$\frac{\partial}{\partial x}v(x,t)\Big|_{x=0} = \sum_{n=0}^{\infty} v_n'(0)t^n = tv(0,t) - 1$$
$$= -1 + \sum_{m=0}^{\infty} \frac{(-1)^m}{2^m(3/2)_m} t^{2m+2}$$

so that

(5.4)
$$v_n'(0) = \begin{cases} 0, & \text{if } n = 2m+1, \\ -1, & \text{if } n = 0, \\ \dfrac{(-1)^m}{2^m(3/2)_m}, & \text{if } n = 2m+2. \end{cases}$$

From (5.1), (5.3) and (5.4) it follows that

$$v_{2m+2}(x) = \frac{(-1)^m}{2^m(3/2)_m} xM(-m-\tfrac{1}{2}, \tfrac{3}{2}; \tfrac{1}{2}x^2),$$
$$v_{2m+1}(x) = \frac{(-1)^m}{2^m(3/2)_m} M(-m-\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}x^2).$$

These give $v_n(x)$ explicitly, and by comparison with (5.2) show that $u_n$ and $v_n$ are linearly independent.

**6. Statement of analogous results.** By a similar method we can find and establish the following.

With the same notation, for

$$u(x,t) = \exp\left[-\frac{xt}{1-t}\right](1-t)^{-\alpha-1}$$

we have

$$u_n(x) = L_n^{(\alpha)}(x),$$

the generalised Laguerre polynomials. Then $u(x,t)$ satisfies the equation

(6.1)
$$x\frac{\partial^2 w}{\partial x^2} + (1+\alpha-x)\frac{\partial w}{\partial x} + t\frac{\partial w}{\partial t} = 0$$

and $u_n(x)$ satisfies

(6.2)
$$xw_n''(x) + (1+\alpha-x)w_n'(x) + nw_n(x) = 0;$$

see [**1**, 22.6.15]. Then

$$v(x,t) = (1-t)^{-\alpha-1}\exp\left[-\frac{xt}{1-t}\right]\left\{\int_0^x e^s s^{-\alpha-1}\,ds \right.$$
$$\left. + x^{-\alpha}e^x\int_0^t (1-s)^{\alpha-1}\exp\left[\frac{xs}{1-s}\right]ds\right\}$$

also satisfies (6.1) and the corresponding $v_n(x)$ satisfy (6.2).

For

$$u(x,t) = (1-t)^{-\alpha}e^{xt}$$

we have

$$u_n(x) = (-1)^n L_n^{(-\alpha-n)}(x).$$

Then $u(x,t)$ satisfies

$$x\frac{\partial^2 w}{\partial x^2} + (1-\alpha-x)\frac{\partial w}{\partial x} + t\frac{\partial w}{\partial t} - t\frac{\partial^2 w}{\partial t \partial x} = 0 \tag{6.3}$$

and $u_n(x)$ satisfies

$$x w_n''(x) + (1-\alpha-n-x)w_n'(x) + n w_n(x) = 0. \tag{6.4}$$

To this we can add that

$$v(x,t) = (1-t)^{-\alpha} e^{xt}\left[-x^{-\alpha}e^{-x}\int_0^x s^{\alpha-1}e^s\,ds + \int_0^t (1-s)^{\alpha-1}e^{-xs}\,ds\right]$$

also satisfies (6.3) and the corresponding $v_n(x)$ satisfy (6.4).

For

$$u(x,t) = (1-xt)^{-\alpha}(1-t)^{-\beta}$$

we have (cf. [**4**, p. 452, Problem 25])

$$u_n(x) = g_n^{(\alpha,\beta)}(x,1) = \frac{(\beta)_m}{m!}F(-n,\alpha;1-\beta-n;x),$$

where $F(a,b;c;z)$ is the hypergeometric function [**1**, p. 556]. Then $u(x,t)$ satisfies

$$x(1-x)\frac{\partial^2 w}{\partial x^2} + [1-\beta-(1+\alpha)x]\frac{\partial w}{\partial x} + \alpha t\frac{\partial w}{\partial t} - (1-x)t\frac{\partial^2 w}{\partial t \partial x} = 0 \tag{6.5}$$

and $u_m(x)$ satisfies

$$x(1-x)u_n''(x) + [1-\beta-n-(1+\alpha-n)x]u_n'(x) + \alpha n u_n(x) = 0. \tag{6.6}$$

To this we can add that

$$v(x,t) = (1-xt)^{-\alpha}(1-t)^{-\beta}\left[-\int_0^x s^{\beta-1}(1-s)^{-\alpha-\beta}\,ds \right. \\ \left. + x^\beta(1-x)^{1-\alpha-\beta}\int_0^t (1-xs)^{\alpha-1}(1-s)^{\beta-1}\,ds\right]$$

also satisfies (6.5) and the corresponding $v_n(x)$ satisfy (6.6).

Finally for

$$u(x,t) = (1-2xt+t^2)^{-\alpha}$$

we have

$$u_n(x) = C_n^{(\alpha)}(x),$$

the Gegenbauer or ultraspherical polynomials. Then $u(x,t)$ satisfies

$$(1-x^2)\frac{\partial^2 w}{\partial x^2} - (2\alpha+1)x\frac{\partial w}{\partial x} + (2\alpha+1)t\frac{\partial w}{\partial t} + t^2\frac{\partial^2 w}{\partial t^2} = 0 \tag{6.7}$$

and $u_n(x)$ satisfies

$$(1-x^2)w_n''(x) - (2\alpha+1)x w_n'(x) + n(2\alpha+n)w_n(x) = 0; \tag{6.8}$$

see [**1**, 22.6.5]. To this we can add that

$$v(x,t) = (1-2xt+t^2)^{-\alpha}\left\{-\int_0^x (1-s^2)^{-\alpha-1/2}\,ds \right. \\ \left. + (1-x^2)^{-\alpha+1/2}\int_0^t (1-2xs+s^2)^{\alpha-1}\,ds\right\}$$

also satisfies (6.7) and the corresponding $v_n(x)$ satisfy (6.8).

**7. Concluding remark.** The technique is somewhat general, as we can try to apply it whenever $u(x, t)$ satisfies a first order homogeneous linear differential equation in $t$, with polynomial coefficients.

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