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SMOOTHNESS OF THE BILLIARD BALL MAP FOR STRICTLY CONVEX DOMAINS NEAR THE BOUNDARY

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ABSTRACT. The billiard ball map in bounded strictly convex domains in $\mathbf{R}^n$ with boundaries of class C^k, $k \geq 2$, is considered and its smoothness of class C^{k-1} up to the boundary is proved.

1. Introduction. It is well known that the billiard ball map for plane billiards in domains bounded by strictly convex smooth curves with everywhere nonzero curvature is smooth up to the boundary (see [**1**]). More precisely, if the boundary is of class C^k, $k \geq 2$, then the billiard ball map is of class C^{k-1} (see [**4**]). This property of such billiards is systematically used in the study of their ergodic properties [**2**, **3**, **4**].

In the present paper we obtain an analogous result for n-dimensional domains. It guarantees the applicability of the theory elaborated in [**5**] (see [**6**]) to billiards in strictly convex bounded regions in $\mathbf{R}^n$ with boundary of class C^k, $k \geq 3$, in particular Pesin's entropy formula [**8**, **9**, **7**] holds for such billiards.

2. Main result. Assume that Ω is a strictly convex bounded domain in $\mathbf{R}^n$ with boundary $\partial\Omega$ of class C^k, $k \geq 2$. Denote by n_z the unit inward normal to $\partial\Omega$ at the point $z \in \Omega$ and define the set

$$\Sigma = \{(z,e) \in \partial\Omega \times S^{n-1}; \langle e, n_z\rangle_n \geq 0\}$$

where $\langle\cdot,\cdot\rangle_n$ is the scalar product in $\mathbf{R}^n$. Then Σ is a compact $(2n-2)$-dimensional manifold of class C^k with boundary $\partial\Sigma$ of class C^{k-1} which is defined by the equation $\langle e, n_z\rangle_n = 0$.

Recall now the definition of the billiard ball map $B\colon \Sigma \to \Sigma$ (see [**4**]). From the convexity of Ω it follows that for each $(z,e) \in \Sigma\backslash\partial\Sigma$ there exists a unique $t > 0$ such that the point

$$z^* = z + te \tag{1}$$

belongs to $\partial\Omega$. For $(z,e) \in \partial\Sigma$ we assume $t = 0$ and $z^* = z$. Then define the map $B : \Sigma \to \Sigma$, $(z,e) \to (z^*, e^*)$, where z^* was defined above and $e^* = s_{z^*}(e)$ where s_{z^*} is the symmetry with respect to the hyperplane $T_{z^*}\partial\Omega$. Obviously B is the identity on $\partial\Sigma$. Moreover, it is well known that B is a diffeomorphism of $\Sigma\backslash\partial\Sigma$ which is continuous up to $\partial\Sigma$. Further on we prove the following theorem.

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THEOREM 1. *Let Ω be a strictly convex bounded domain in $\mathbf{R}^n$ with boundary of class C^k, $k \geq 2$. Then the billiard ball map B is a diffeomorphism of class C^{k-1} of the compact manifold Σ onto itself.*

3. Proof of Theorem 1. Take an arbitrary point $M \in \partial\Omega$. From the strict convexity of Ω it follows that there exists a neighbourhood U_M of M such that $\partial\Omega \cap U_M$ can be represented by the equation

$$y = g(x) \tag{2}$$

where $g(x)$ is a function of class C^k defined in an $(n-1)$-dimensional neighbourhood of the origin such that $g(0) = 0$, $\partial g(0) = 0$ and the matrix $\partial^2 g(0)$ is positively definite, i.e. there exists a constant $C > 0$ such that $\langle \partial^2 g(0)\xi, \xi\rangle \geq 2C|\xi|^2$ for all $\xi \in \mathbf{R}^{n-1}$. Here $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbf{R}^{n-1}$. Choose a convex neighbourhood V of 0 in $\mathbf{R}^{n-1}$ such that for all $x \in V$ we have $z = (x, g(x)) \in \partial\Omega \cap U_M$,

$$|\partial g(x)| < \min((15/n)^{1/2}/4, 15^{-1/2}) \tag{3}$$

and

$$\langle \partial^2 g(x)\xi, \xi\rangle \geq C|\xi|^2 \tag{4}$$

for all $\xi \in \mathbf{R}^{n-1}$. Let V' be a neighbourhood of 0 in $\mathbf{R}^{n-1}$ such that $\overline{V'} \subset V$ and denote $U'_M = \{(x, g(x)); x \in V'\}$. From the covering $\{U'_M\}_{M \in \partial\Omega}$ of $\partial\Omega$ we can choose a finite subcovering and further on all considerations will be in one of these neighbourhoods denoted by U' such that $\overline{U'} \subset U = \{(x, g(x)); x \in V\}$.

The unit inward normal at the point $z = (x, g(x)) \in U$ is given by

$$n_z = (-\partial g(x), 1)/(1 + |\partial g(x)|)^2)^{1/2}. \tag{5}$$

For any vector $e \in S^{n-1}$ such that $(z, e) \in \Sigma$ put $e = (\xi, \eta) = (\xi_1, \ldots, \xi_{n-1}, \eta) \in \mathbf{R}^{n-1}_\xi \times \mathbf{R}_\eta$ and denote

$$\varepsilon = \langle e, n_z \rangle_n. \tag{6}$$

Here ε is a nonnegative number vanishing only on $\partial\Sigma$. From (5) and (6) we obtain

$$\eta = \varepsilon(1 + |\partial g(x)|^2)^{1/2} + \langle \partial g(x), \xi \rangle. \tag{7}$$

Since we investigate the billiard ball map near $\partial\Sigma$, we can assume $0 \leq \varepsilon < 1/4$. Then from (7) and (3) it follows that

$$\begin{aligned} |\eta| &\leq \varepsilon(1 + |\partial g(x)|^2)^{1/2} + |\partial g(x)|\,|\xi| \leq (1 + 1/15)^{1/2}/4 + 15^{-1/2}|\xi| \\ &= 15^{-1/2}(1 + |\xi|). \end{aligned} \tag{8}$$

Since $e = (\xi, \eta) \in S^{n-1}$, then $1 = |\xi|^2 + |\eta|^2 \leq |\xi|^2 + (1 + |\xi|)^2/15$ which implies $8|\xi|^2 + |\xi| - 7 \geq 0$ or $|\xi| \geq 7/8$.

Denote

$$W_j = \Big\{(x, g(x), \xi, \eta) \in U \times S^{n-1}; x \in V, |\xi_j| > 3n^{-1/2}/4\ ,$$
$$\eta = \varepsilon(1 + |\partial g(x)|^2)^{1/2} + \langle \partial g(x), \xi\rangle, 0 \leq \varepsilon < 1/4\Big\}, \quad j = 1, 2, \ldots, n-1,$$

$$W'_j = \Big\{(x, g(x), \xi, \eta) \in U' \times S^{n-1}; x \in V', |\xi_j| > 7n^{-1/2}/8\ ,$$
$$\eta = \varepsilon(1 + |\partial g(x)|^2)^{1/2} + \langle \partial g(x), \xi\rangle,\ 0 \leq \varepsilon < \varepsilon_0\Big\}, \quad j = 1, 2, \ldots, n-1,$$

where the constant $\varepsilon_0 < 1/4$ will be determined below. Obviously

$$U \times \{e \in S^{n-1}; 0 \le \langle e, n_z\rangle_n < 1/4\} = \bigcup_{j=1}^{n-1} W_j,$$

$$U' \times \{e \in S^{n-1}; 0 \le \langle e, n_z\rangle_n < \varepsilon_0\} = \bigcup_{j=1}^{n-1} W_j'.$$

Thus the union of all W_j' (or W_j) related to a finite covering $\{U'\}$ of $\partial\Omega$ (or to the respective convex neighbourhoods $U \supset \overline{U'}$) provides a finite covering of a neighbourhood of $\partial\Sigma$ in Σ.

Below we fix an index $j : 1 \le j \le n-1$, and denote

$$\phi(x, \varepsilon, \xi) = |\xi|^2 + (\varepsilon(1 + |\partial g(x)|^2)^{1/2} + \langle \partial g(x), \xi\rangle)^2 - 1.$$

Then the partial derivative $\phi_{\xi_j}(x, \varepsilon, \xi)$ equals

$$2\xi_j + 2(\varepsilon(1 + |\partial g(x)|^2)^{1/2} + \langle \partial g(x), \xi\rangle) g_{x_j}(x)$$

and for $(z, e) \in W_j$ we have

$$\begin{aligned} |\phi_{\xi_j}(x, \varepsilon, \xi)/2| &\ge |\xi_j| - |\varepsilon(1 + |\partial g(x)|^2)^{1/2} + \langle \partial g(x), \xi\rangle||g_{x_j}(x)| \\ &\ge 3n^{-1/2}/4 - 15^{-1/2}(1 + |\xi|)(15/n)^{1/2}/4 \\ &\ge 3n^{-1/2}/4 - 2n^{-1/2}/4 = n^{-1/2}/4 > 0. \end{aligned}$$

By the implicit function theorem there exists $\xi_j(x, \xi_1, \dots, \xi_{j-1}, \xi_{j+1}, \dots, \xi_{n-1}, \varepsilon)$ a function of class C^{k-1} satisfying $\phi(x, \varepsilon, \xi) = 0$. Thus in W_j and W_j' we can use the local coordinates $(x, \xi_1, \dots, \xi_{j-1}, \xi_{j+1}, \dots, \xi_{n-1}, \varepsilon)$ and in any of these neighbourhoods the boundary $\partial\Sigma$ is given by $\varepsilon = 0$.

From $B(z, e) = (z, e)$ on $\partial\Sigma$ where $\varepsilon = 0$, the continuity of B up to the boundary and the fact that $\overline{U'} \subset U$ it follows that there exists a positive number $\varepsilon_0 < 1/4$ such that for all $z \in U'$, $e \in S^{n-1}$: $0 < \langle e, n_z\rangle_n < \varepsilon_0$ we have $z^* \in U$. Here z^* is the point determined by (1).

Now assume that $(z, e) \in W_j'$. In the local coordinates chosen above (1) can be written as

$$x^* = x + t\xi, \tag{9}$$

$$g(x^*) = g(x) + t\eta. \tag{10}$$

In view of (7) last equality becomes

$$g(x + t\xi) = g(x) + t(\varepsilon(1 + |\partial g(x)|^2)^{1/2} + \langle \partial g(x), \xi\rangle). \tag{11}$$

By Taylor's formula we have

$$g(x + t\xi) = g(x) + t\int_0^1 \langle \partial g(x + st\xi), \xi\rangle\, ds. \tag{12}$$

Thus from (11) we obtain

$$\int_0^1 \langle \partial g(x + st\xi), \xi\rangle\, ds = \varepsilon(1 + |\partial g(x)|^2)^{1/2} + \langle \partial g(x), \xi\rangle. \tag{13}$$

Since the derivative of the left-hand side with respect to t is

$$\int_0^1 s\langle\partial^2 g(x+st\xi)\xi,\xi\rangle\,ds \geq \int_0^1 sC|\xi|^2\,ds \geq C\left(\frac{7}{8}\right)^2/2 = \frac{49C}{128} > 0,$$

from the implicit function theorem it follows that t is a function of (x,ξ,ε) of class C^{k-1}, hence by (9) x^* is a function of (x,ξ,ε) of class C^{k-1} for $0 \leq \varepsilon < \varepsilon_0$. In order to express more explicitly the dependence of t and x^* on ε, replace (12) by the expansion

$$g(x+t\xi) = g(x) + t\langle\partial g(x),\xi\rangle + t^2\int_0^1 (1-s)\langle\partial^2 g(x+st\xi)\xi,\xi\rangle\,ds.$$

Hence t satisfies the equation

$$t\int_0^1 (1-s)\langle\partial^2 g(x+st\xi)\xi,\xi\rangle\,ds = \varepsilon(1+|\partial g(x)|^2)^{1/2}. \tag{14}$$

This shows that $t = O(\varepsilon)$, $x^* = x + O(\varepsilon)$ uniformly on $(z,e) \in W'_j$.

Now we shall express

$$\begin{aligned}\varepsilon^* &= -\langle n_{z^*}, e\rangle_n\\ &= (\langle\partial g(x^*),\xi\rangle - \varepsilon(1+|\partial g(x)|^2)^{1/2} - \langle\partial g(x),\xi\rangle)/(1+|\partial g(x^*)|^2)^{1/2}.\end{aligned}$$

This representation allows us to conclude that ε^* is a function of (x,ξ,ε) of class C^{k-1}. By Taylor's formula

$$\partial g(x^*) = \partial g(x+t\xi) = \partial g(x) + t\int_0^1 \partial^2 g(x+st\xi)\xi\,ds$$

and by (14) we obtain

$$\varepsilon^* = \varepsilon\left(\frac{1+|\partial g(x)|^2}{1+|\partial g(x^*)|^2}\right)^{1/2}\frac{\int_0^1 s\langle\partial^2 g(x+st\xi)\xi,\xi\rangle\,ds}{\int_0^1 (1-s)\langle\partial^2 g(x+st\xi)\xi,\xi\rangle\,ds} \tag{15}$$

hence $\varepsilon^* = \varepsilon + O(\varepsilon^2)$ uniformly on $(z,e) \in W'_j$. This implies that for ε_0 small enough we shall have $\varepsilon^* < 1/4$ for any $(z,e) \in W'_j$.

Further on, we have $e^* = e - 2\varepsilon^* n_{z^*}$, hence ξ^* is a function of (x,ξ,ε) of class C^{k-1}. Write the equality of the first $n-1$ coordinates of e^*:

$$\xi^* = \xi + 2\varepsilon\frac{(1+|\partial g(x)|^2)^{1/2}}{1+|\partial g(x^*)|^2}\frac{\int_0^1 s\langle\partial^2 g(x+st\xi)\xi,\xi\rangle\,ds}{\int_0^1 (1-s)\langle\partial^2 g(x+st\xi)\xi,\xi\rangle\,ds}\partial g(x^*). \tag{16}$$

Last equality shows that $\xi^* = \xi + O(\varepsilon)$ uniformly on $(z,e) \in W'_j$. This implies that for ε_0 small enough we shall have $|\xi^*_j| > 3n^{-1/2}/4$, hence $(z^*,e^*) \in W_j$.

Thus we have shown that B is a map of class C^{k-1} from W'_j into W_j. Since W'_j is a sufficiently small neighbourhood of an arbitrary point of $\partial\Sigma$, this proves the smoothness of B near the boundary.

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