ABSTRACT. Let \( v \) be a potential in the unit ball of \( \mathbb{R}^n \), and \( \mathcal{M}_p(v; r) \) be its \( p \)th order mean over the sphere of radius \( r \) centred at the origin. It is shown that, as \( r \to 1^- \), the function \( f(r) = (1 - r)^{(n-1)/(n-2)} \mathcal{M}_p(v; r) \) has limit 0 when \( 1 \leq p < (n-1)/(n-2) \), and has lower limit 0 when \( n \geq 3 \) and \( (n-1)/(n-2) \leq p < (n-1)/(n-3) \). This extends a result of Stoll, who showed that, when \( n = 2 \) and \( p = +\infty \), \( \lim\inf_{r \to 1^-} f(r) = 0 \). Examples are given to show that the theorems presented are best possible.

1. Introduction and results.

THEOREM A. If \( v \) is a (Green) potential in the unit disc, then
\[
\liminf_{r \to 1^-} \sup_{0 \leq \theta < 2\pi} v(re^{i\theta}) = 0.
\]

In the special case where \( v = -\log |B| \) and \( B \) is a convergent Blaschke product in the unit disc, Theorem A is due to Heins [1] (see also [3]). The result for general potentials was only recently established by Stoll [4] (see also [2]). It does not have an obvious analogue for potentials in the unit ball of \( \mathbb{R}^n \) \( (n \geq 3) \), for such functions can be valued identically \( +\infty \) on a given radius.

We denote an arbitrary point of Euclidean space \( \mathbb{R}^n \) \( (n \geq 2) \) by \( X = (x_1, \ldots, x_n) \) and put \( |X| = (x_1^2 + \cdots + x_n^2)^{1/2} \). Let
\[
B(r) = \{X \in \mathbb{R}^n : |X| < r\}, \quad S(r) = \{X \in \mathbb{R}^n : |X| = r\},
\]
\[
A(r_1, r_2) = \{X \in \mathbb{R}^n : r_1 < |X| < r_2\},
\]
and let \( \sigma \) denote normalized surface area measure on \( S(r) \). If \( f \) is a nonnegative measurable function on \( S(r) \), define
\[
\mathcal{M}_p(f; r) = \left\{ \int_{S(r)} f^p d\sigma \right\}^{1/p} \quad (p > 0)
\]
and
\[
\mathcal{M}_\infty(f; r) = \sup \{f(X) : X \in S(r)\}.
\]

As will be seen below, Theorem A has the following analogue in higher dimensions:
\[
\liminf_{r \to 1^-} (1 - r)^{-(n-1)/(n-2)} \mathcal{M}_\infty(v; r) = 0
\]
for any potential \( v \) in \( B(1) \). However, recalling the well-known property that 
\[ M_1(v; r) \to 0 \quad \text{as} \quad r \to 1^- \], it is natural to consider the limiting behaviour of \( M_p(v; \cdot) \) for other values of \( p \). This leads to the following results.

**Theorem 1.** If \( v \) is a potential in \( B(1) \subset \mathbb{R}^n \) (\( n \geq 2 \)), and \( 1 < p < (n - 1)/(n - 2) \), then

\[ (1 - r)^{(n-1)(1-1/p)}M_p(v; r) \to 0 \quad (r \to 1^-). \]

**Theorem 2.** If \( v \) is a potential in \( B(1) \subset \mathbb{R}^n \) (\( n \geq 3 \)), and \( (n - 1)/(n - 2) < p < (n - 1)/(n - 3) \), then

\[ \liminf_{r \to 1^-} (1 - r)^{(n-1)(1-1/p)}M_p(v; r) = 0. \]

In later sections of the paper we prove Theorems 1 and 2 using ideas from another paper by Stoll [5]. First we mention some examples to show that these results are best possible.

**Example 1.** If \( 0 < p < 1 \) and \( v \) is a potential in \( B(1) \subset \mathbb{R}^n \) (\( n \geq 2 \)), it is a simple consequence of Jensen's inequality that 
\[ JKPiv; r) \to 0 \quad \text{as} \quad r \to 1^- \]. However, neither (1) nor (2) hold. To see this, let \( \beta = \min\{1, (n - 1)(1/p - 1)\} \). Since \( (1 - |X|) \) is a potential in \( B(1) \) and \( \beta \in (0,1] \), it follows easily that \( v_0(X) = (1 - |X|)^\beta \) is also a potential in \( B(1) \), but

\[ (1 - r)^{(n-1)(1-1/p)}M_p(v_0; r) \geq 1 \quad (0 < r < 1). \]

**Example 2.** If \( n \geq 3 \), there is a potential \( v \) in \( B(1) \) such that \( M_p(v; r) \equiv +\infty \) for all \( p > (n - 1)/(n - 3) \). In fact, if \( n = 3 \), let \( u(X) = -\log(x_2^2 + x_3^2) \), let \( h \) be the greatest harmonic minorant of \( u \) in \( B(1) \), and define \( v = u - h \). Then \( v \) is a potential and it is easy to see that \( M_\infty(v; r) \equiv +\infty \). If \( n \geq 4 \), let \( u(X) = (x_2^2 + \cdots + x_n^2)^{(3-n)/2} \), and define the corresponding potential \( v \) as before. Straightforward estimates show that \( M_p(v; r) \equiv +\infty \) for \( p > (n - 1)/(n - 3) \).

It is also easy to see that, if \( (n - 1)/(n - 2) \leq p < (n - 1)/(n - 3) \) and the measure associated with a potential \( v \) in \( B(1) \) comprises point masses arbitrarily close to \( S(1) \), then

\[ \limsup_{r \to 1^-} (1 - r)^{(n-1)(1-1/p)}M_p(v; r) = +\infty. \]

**Example 3.** Theorems 1 and 2 fail if we replace "potential" by "positive superharmonic function". To see this, let \( h \) be the positive harmonic function given by

\[ h(X) = (1 - |X|^2)||X - (1,0,\ldots,0)||^{-n} \quad (X \in B(1)), \]

and let \( p \geq 1 \). Then \( (1 - r)^{(n-1)(1-1/p)}M_p(h; r) \) has a finite positive limit as \( r \to 1^- \). However, Theorems 1 and 2 do hold for positive superharmonic functions which do not majorize any positive multiple of a Poisson kernel. Details are given in §§5.1, 5.2.

**Example 4.** If \( \varepsilon > 0 \) and \( p \geq 1 \), then there is a potential \( v \) in \( B(1) \) such that

\[ (1 - r)^{(n-1)(1-1/p)} M_p(v; r) \to +\infty \quad (r \to 1^-). \]

Details may be found in §5.3.
2. Two preliminary lemmas.

2.1. Let \( O \) denote the origin of \( \mathbb{R}^n \) and \( X^* \) be the image of a point \( X \) under inversion of centre \( O \) and radius 1. We use \( G(\cdot, \cdot) \) to denote the Green kernel of \( B(1) \) so that, in the case \( n = 2 \),

\[
G(X, Y) = \begin{cases} 
- \log |Y - X| + \log(|X| \cdot |Y - X^*|) & (X \neq O), \\
- \log |Y| & (X = O),
\end{cases}
\]

and in the case \( n \geq 3 \),

\[
G(X, Y) = \begin{cases} 
|Y - X|^{2-n} - (|X| \cdot |Y - X^*|)^{2-n} & (X \neq O), \\
|Y|^2 - n - 1 & (X = O).
\end{cases}
\]

We recall that there is a one-to-one correspondence between potentials \( v \) on \( B(1) \) and measures \( \mu \) on \( B(1) \) which satisfy

\[
(3) \quad \int_{B(1)} (1 - |Y|) \, d\mu(Y) < +\infty.
\]

For the remainder of §2, we will assume that either \( n = 2 \) and \( p > 1 \), or \( n \geq 3 \) and \( 1 < p < (n - 1)/(n - 3) \). It will be convenient to let \( \alpha = (n - 1)(1 - p^{-1}) \), which clearly lies in the interval \((0, 1)\) when \( n = 2 \), and \((0, 2)\) when \( n \geq 3 \). Also, \( C(a, b, c, \ldots) \) will denote a positive constant depending at most on \( a, b, c, \ldots \), not necessarily the same on any two occurrences.

2.2. Let \( E(r) = A((5r-1)/4, (3r+1)/4) \) so that, if \( y \in B(1) \setminus E(r) \) and \( X \in S(r) \), then \(|Y - X| \geq ||Y| - r| > (1 - r)/4\).

**Lemmas 1.** If \( p, \alpha \) are as above and \( \mu \) is a measure on \( B(1) \) satisfying (3), then

\[
(1 - r)^\alpha \int_{B(1) \setminus E(r)} \mathcal{M}_p(G(\cdot, Y); r) \, d\mu(Y) \to 0 \quad (r \to 1^-).
\]

To see this, let \(|X| = r \) and \( Y \in B(1) \setminus E(r) \). If \( n \geq 3 \), then

\[
G(X, Y) = \begin{cases} 
|Y - X| \cdot |rY - r^{-1}X|^{2-n} \cdot |rY - r^{-1}X|^{n-2} - |Y - X|^{n-2} \\
2^{-1}(n-2)|Y - X|^{-n} \cdot |rY - r^{-1}X|^2 - |Y - X|^2
\end{cases}
\]

\[
\leq 2^{\alpha-1}(n-2)(1-r)^{-\alpha}|Y - X|^{\alpha-n}|rY - r^{-1}X|^2 - |Y - X|^2.
\]

If \( n = 2 \), then

\[
G(X, Y) = \log(|rY - r^{-1}X|/|Y - X|)
\]

\[
\leq |Y - X|^{-1} \cdot |rY - r^{-1}X| - |Y - X|)
\]

\[
\leq 2^{2\alpha-1}(1-r)^{-\alpha}|Y - X|^{\alpha-2}|rY - r^{-1}X|^2 - |Y - X|^2.
\]

Thus, letting \( \rho = |Y| \) we have, for any \( n \geq 2 \),

\[
\int_{S(r)} [G(X, Y)]^p \, d\sigma(X) \leq C(n, p)(1 - r)^{-\alpha p}
\]

\[
\times \int_0^\pi \sin^{n-2} \theta \{(\rho - r \cos \theta)^2 + (r \sin \theta)^2\}^{(1-n-p)/2} \{(1 - r^2)(1 - \rho^2)\}^p \, d\theta
\]

\[
= C(n, p)(1 - r)^{-\alpha p} \{(1 - r^2)(1 - \rho^2)\}^p
\]

\[
\times \int_0^\infty t^{n-2}(1 + t^2)^{(p+1-n)/2} \{(\rho - r)^2 + (\rho + r)^2\}^{(1-n-p)/2} \, dt,
\]

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using the substitution \( t = \tan(\theta/2) \). Hence, if \( \kappa = |\rho - r|/(\rho + r) \) and \( r > \frac{1}{2} \),
\[
(1 - r)^\alpha \mathcal{M}_p(G(\cdot, Y); r)^p
\leq C(n, p) \frac{(1 - r)(1 - \rho)}{(1 - r)(1 - \rho)/\kappa)^p}
\int_0^\infty t^{n-2}(1 + t^2)^{(p+1-n)/2}(\kappa^2 + t^2)^{(1-n-p)/2} dt
= C(n, p) \frac{(1 - r)(1 - \rho)/\kappa)^p}{(1 - r)(1 - \rho)}
\times \int_0^\infty x^{n-2}(1 + x^2)^{(p+1-n)/2}(1 + x^2)^{(1-n-p)/2} dx
\leq C(n, p) \frac{(1 - r)(1 - \rho)/\kappa)^p}{(1 - r)(1 - \rho)}
\int_0^\infty x^{n-2}(1 + x^2)^{(1-n-p+(p+1-n)^+)/2} dx
\leq C(n, p) \frac{(1 - r)(1 - \rho)/|\rho - r|)^p}{(1 - r)(1 - \rho)}
\]
by means of the substitution \( t = /ex \). Thus
\[
(1 - r)^\alpha \mathcal{M}_p(G(\cdot, Y); r) \leq C(n, p)(1 - |Y|) \quad (Y \in B(1) \setminus E(r)).
\]

Now let \( \varepsilon > 0 \) and choose \( R \in (0, 1) \) large enough to ensure that
\[
\int_{A(R, 1)} (1 - |Y|) d\mu(Y) < \varepsilon.
\]
Simple estimates yield
\[
G(X, Y) \leq C(n, R)(1 - |X|) \quad (|Y| \leq R; (1 + R)/2 < |X| < 1).
\]
Hence, if \( r \in ((1 + R)/2, 1) \), it follows from (4) that
\[
(1 - r)^\alpha \int_{B(1) \setminus E(r)} \mathcal{M}_p(G(\cdot, Y); r) d\mu(Y)
\leq C(n, R)(1 - r)^{\alpha+1} \mu(|Y| \leq R) + C(n, p)\varepsilon
\rightarrow C(n, p)\varepsilon \quad (r \rightarrow 1-).
\]
Since \( \varepsilon > 0 \) was arbitrary, the lemma is proved.

2.3.

**Lemma 2.** Let \( p, \alpha \) be as above, let \( r > \frac{1}{2} \) and \( Y \in E(r) \). Then
\[
(1 - |Y|)^{-1}(1 - r)^\alpha \mathcal{M}_p(G(\cdot, Y); r) \leq \begin{cases} 
C(n, p) & (0 < \alpha < 1), \\
C(n, p) \log ((1 - r)/|Y| - r) & (\alpha = 1), \\
C(n, p) [(1 - r)/|Y| - r]|^{\alpha - 1} & (1 < \alpha < 2).
\end{cases}
\]
To prove the lemma, let \( r > \frac{1}{2}, X \in S(r), Y \in E(r) \) and \( \rho = |Y| \). Then
\[
|rY - r^{-1}X| \leq |rY - Y| + |Y - X| + |X - r^{-1}X| < 2(1 - r) + |Y - X|
\]
and
\[
|\rho - r| < (1 - r)/4 < (1 - \rho)/3.
\]
We deal first with the case \( n \geq 3 \), where
\[
G(X, Y) \leq (n - 2)|Y - X|^{2-n}\{1 - |Y - X|/|rY - r^{-1}X|\}
\leq 2(n - 2)(1 - r)|Y - X|^{2-n}/\{2(1 - r) + |Y - X|\}.
\]
Hence, as in the proof of Lemma 1,

\[
\int_{S(r)} [G(X,Y)]^p \, d\sigma(X) \\
\leq C(n,p)(1-r)^p \int_0^{\pi} \sin^{n-2} \theta \{ (\rho - r \cos \theta)^2 + (r \sin \theta)^2 \}^{p-np/2} \\
\times \{ 2(1-r) + [(\rho - r \cos \theta)^2 + (r \sin \theta)^2]^{1/2} \}^{-p} \, d\theta \\
\leq C(n,p)(1-r)^p \int_0^{\infty} t^{n-2} (1+\kappa^2)^{(n-1)(p-2)/2} (\kappa^2 + t^2)^{p-np/2} \\
\times \{ (1-r)(1+t^2)^{1/2} + (\kappa^2 + t^2)^{1/2} \}^{-p} \, dt \\
\leq C(n,p)(1-r)^p \int_0^{\infty} t^{n-2} (1+\kappa^2)^{(n-1)(p-2)/2} \\
\times (\kappa^2 + t^2)^{p-np/2} \{ (1-r) + t \}^{-p} \, dt
\]

where \( \kappa = |\rho - r|/(\rho + r) \). We split up the integral in (5) into the components \( J_1, J_2, J_3, J_4 \) corresponding to the intervals \([0,\kappa], [\kappa, 1-r], [1-r, 1], [1, \infty) \). Then

\[
J_1 \leq C(n,p)(1-r)^{-p} \int_0^{\kappa} t^{n-2}(\kappa^2 + t^2)^{p-np/2} \, dt \\
= C(n,p)(1-r)^{-p} \kappa^{p(1-\alpha)} \int_0^{\pi/4} \sin^{-2} \phi \cos^{p-2} \phi \, d\phi
\]

using the substitution \( t = \kappa \tan \phi \). Next

\[
J_2 \leq C(n,p)(1-r)^{-p} \int_0^{1-r} t^{n-2}(1-t)^{p-1} \, dt \\
= C(n,p)(1-r)^{-p} \int_0^{1-r} t^{-p(\alpha-1)-1} \, dt \\
= \begin{cases} 
C(n,p)(1-r)^{-p\alpha} & (0 < \alpha < 1), \\
C(n,p)(1-r)^{-p} \log \{(1-r)/\kappa \} & (\alpha = 1), \\
C(n,p)(1-r)^{-p\kappa^{p(1-\alpha)}} & (1 < \alpha < 2), 
\end{cases}
\]

\[
J_3 \leq C(n,p) \int_{1-r}^{1} t^{n-2}(1-t)^{p-1} (1-r+t)^{-p} \, dt \\
\leq C(n,p) \int_{1-r}^{1} (1-r+t)^{p+n-p-2} \, dt \\
\leq C(n,p)(1-r)^{-\alpha p},
\]

\[
J_4 \leq C(n,p) \int_1^{\infty} t^{-n} \, dt = C(n,p).
\]

Hence, from (5),

\[
\int_{S(r)} [G(X,Y)]^p \, d\sigma(X) \leq \begin{cases} 
C(n,p)(1-r)^{p(1-\alpha)} & (0 < \alpha < 1), \\
C(n,p) \log \{(1-r)/|\rho - r| \} & (\alpha = 1), \\
C(n,p)|\rho - r|^{p(1-\alpha)} & (1 < \alpha < 2). 
\end{cases}
\]

The \( n \geq 3 \) case of the lemma now follows easily on taking \( p \)th roots and noting that \( \log \{(1-r)/|\rho - r| \} > \log 4 > 1 \).
The \( n = 2 \) case of the lemma is more straightforward, as here we have \( 0 < \alpha < 1 \).

Since

\[ G(X, Y) \leq \log\{1 + 2(1 - r)/|Y - X|\}, \]

it follows as before that

\[
\int_{S(r)} [G(X, Y)]^p \, d\sigma(X)
\]

\[
\leq \pi^{-1} \int_0^\pi \left\{ \log\left[ 1 + 2(1 - r)\{(\rho - r \cos \theta)^2 + (r \sin \theta)^2\}^{-1/2} \right] \right\}^p \, d\theta
\]

\[
\leq 2\pi^{-1} \int_0^\infty (1 + t^2)^{-1} \left\{ \log\left[ 1 + 4(1 - r)(1 + t^2)^{1/2}(\kappa^2 + t^2)^{-1/2} \right] \right\}^p \, dt.
\]

We split up the integral in (6) into the components \( J_1, J_2, J_3 \) corresponding to the intervals \([0, 1 - r], [1 - r, 1], [1, \infty)\). Then

\[ J_1 \leq \int_0^{1-r} \{\log [7(1 - r)t^{-1}]\}^p \, dt = C(p)(1 - r), \]

\[ J_2 \leq \int_{1-r}^1 \{\log [1 + 6(1 - r)t^{-1}]\}^p \, dt \]

\[ \leq 6p(1 - r)^p \int_{1-r}^1 t^{-p} \, dt \leq C(p)(1 - r), \]

\[ J_3 \leq \int_1^\infty t^{-2}\{\log [1 + 6(1 - r)]\}^p \, dt \leq 6p(1 - r)^p, \]

and so from (6)

\[ \int_{S(r)} [G(X, Y)]^p \, d\sigma(X) \leq C(p)(1 - r). \]

Hence

\[ (1 - |Y|)^{-1}(1 - r)^\alpha \mathcal{M}_p(G(, Y); r) \leq \frac{2}{3}(1 - r)^{-1/p} \mathcal{M}_p(G(, Y); r) \leq C(p), \]

as required.

3. **Proof of Theorem 1.** Let \( 1 < p < (n - 1)/(n - 2) \), so that \( 0 < \alpha < 1 \), and let \( \mu \) be the measure corresponding to the potential \( v \). By Minkowski’s inequality,

\[
\mathcal{M}_p(v; r) = \left\{ \int_{S(r)} \left[ \int_{B(1)} G(X, Y) \, d\mu(Y) \right]^p d\sigma(X) \right\}^{1/p}
\]

\[ \leq \int_{B(1)} \mathcal{M}_p(G(, Y); r) \, d\mu(Y). \]

Thus, from Lemmas 1 and 2,

\[ (1 - r)^\alpha \mathcal{M}_p(v; r) \leq o(1) + C(n, p) \int_{E(r)} (1 - |Y|) d\mu(Y) = o(1) \quad (r \to 1) \]

in the light of (3).
4. Proof of Theorem 2.

4.1. If \( \mu \) is a measure on \( B(1) \) satisfying (3), we define a finite measure \( \mu^* \) on \( B(1) \) by \( d\mu^*(Y) = (1 - |Y|) d\mu(Y) \). For any interval \( I \subseteq [0, 1) \), let \( A(I) = \{ X : |X| \in I \} \). The maximal function \( M(d\mu^*) \) is defined by

\[
M(d\mu^*)(t) = \sup_{I \ni t} \mu^*(A(I))/|I|,
\]

where \( |I| \) denotes the Lebesgue measure of \( I \). Also, for each \( k \in \mathbb{N} \), we define the interval \( I_k = [1 - 2^{-k}, 1 - 2^{-k-1}] \), and denote by \( \mu_k^* \) the restriction of \( \mu^* \) to \( A(I_k) \). Well-known estimates for the maximal function yield the following lemma (cf. [5, p. 453]).

**Lemma A.** For each \( k \in \mathbb{N} \), there exists \( r_k \in (1 - 2^{-k}/5, 1 - 2^{-k}/3) \) such that

\[
M(d\mu_k^*)(r_k) < 2^{k+4} \mu^*(A(I_k)).
\]

4.2. We now prove Theorem 2. Let \( n \geq 3 \) and \( (n-1)/(n-2) < p < (n-1)/(n-3) \), so that \( 1 < \alpha < 2 \), and let \( \mu \) be the measure corresponding to the potential \( v \). We give below the argument for \( 1 < \alpha < 2 \), the case \( \alpha = 1 \) being similar.

As in §3,

\[
(1 - r)^\alpha M_p(v; r) \leq o(1) + C(n, p) \int_{E(r)} \left[ \frac{1 - r}{|Y| - r} \right]^{\alpha - 1} d\mu^*(Y)
\]

as \( r \to 1^- \). Now let \((r_k)\) be a sequence as in Lemma A. Fixing \( k \) and approximating \([1 - r_k]/|x - r_k|\)^{\alpha - 1} by a monotone increasing sequence of step functions of the form

\[
f(x) = \sum_{j=1}^N a_j \chi_{I_j}(x),
\]

where each \( a_j \) is nonnegative and each interval \( I_j \) is symmetric about \( r_k \) it follows that

\[
\int_{E(r_k)} \left[ \frac{1 - r_k}{|Y| - r_k} \right]^{\alpha - 1} d\mu^*(Y) \leq 2M(d\mu_k^*)(r_k)(1 - r_k)^{\alpha - 1} \int_0^{(1-r_k)/4} x^{1-\alpha} \, dx
\]

\[
= C(n, p)(1 - r_k)M(d\mu_k^*)(r_k).
\]

Since, from Lemma A,

\[
(1 - r_k)M(d\mu_k^*)(r_k) \leq 2^{4} \mu^*(A(I_k)),
\]

it now follows from (7) that

\[
(1 - r_k)^\alpha M_p(v; r_k) \leq o(1) + C(n, p)\mu^*(A(I_k)) = o(1) \quad (k \to \infty),
\]

as required.

5. Examples 3 and 4.

5.1. Let \( h \) be as in Example 3 and \( p \geq 1 \). Then

\[
\int_{S(r)} [h(X)]^p \, d\sigma(X)
\]

\[
= C(n)(1 - r^2)^p \int_0^\pi \sin^{n-2} \theta \{ (1 - r \cos \theta)^2 + (r \sin \theta)^2 \}^{-np/2} \, d\theta
\]

\[
= C(n)(1 - r^2)^p \int_0^\infty t^{n-2}(1 + t^2)^{-n+np/2} \{ (1 - r)^2 + (1 + r)^2 t^2 \}^{-np/2} \, dt.
\]
Splitting up the integral in (8) into the components $J_1, J_2$ corresponding to the intervals $[0, 1], [1, \infty)$, it is clear from dominated convergence that $J_2(r)$ has a finite limit as $r \to 1^-$. Also

$$(1 - r)^{np + 1 - n} J_1(r) = (1 + r)^{1 - n} \int_0^{\tan^{-1} \left( \frac{(1 + r)/(1 - r)}{} \right)} \sin^{n-2} \phi \cos^{p-1} \phi$$

$$\times \left( 1 + (1 - r)^2 (1 + r)^{-2} \tan^2 \phi \right)^{1 - n + np/2} \, d\phi$$

has a finite positive limit as $r \to 1^-$, so

$$(9) \quad (1 - r)^{(n - 1)(1 - 1/p)} \mathcal{M}_p(h; r) \to C(n, p) \quad (r \to 1^-).$$

5.2. By Minkowski's inequality it is sufficient to show that

$$(10) \quad (1 - r)^{(n - 1)(1 - 1/p)} \mathcal{M}_p(h; r) \to 0 \quad (r \to 1^-)$$

for any positive harmonic function $h$ in $B(1)$ which does not majorize any positive multiple of a Poisson kernel. For simplicity we show this when $n = 2$; the higher dimensional argument requires minor modification.

Let $\mu$ be the measure on $[0, 2\pi)$ associated with $h$ in the Poisson integral representation. From §5.1 it is straightforward to deduce that

$$\limsup_{r \to 1^-} (1 - r)^{(p - 1)} \int_{\alpha}^{\beta} \left\{ h(re^{i\theta}) \right\}^p d\theta \leq C(p) \left\{ \mu(\{\alpha, \beta\}) \right\}^p$$

for $0 \leq \alpha < \beta < 2\pi$. Let $k$ be a positive integer. No singleton has positive measure, so we can let $0 = \gamma_0 < \gamma_1 < \cdots < \gamma_k = 2\pi$ be such that

$$\mu(\{\gamma_i - \gamma_{i-1}\}) = \mu([0, 2\pi))/k$$

for each $i \in \{1, \ldots, k\}$. Then

$$\limsup_{r \to 1^-} (1 - r)^{(p - 1)} \int_0^{2\pi} \left\{ h(re^{i\theta}) \right\}^p d\theta \leq C(p) \left\{ \mu([0, 2\pi]) \right\}^p k^{1 - p}.$$

Since $k$ can be arbitrarily large, (10) holds for any $p > 1$.

5.3. To establish Example 4, let $\varepsilon > 0$ and $p \geq 1$, and choose $\beta \in (1 - \varepsilon/n, 1)$. If $h$ is as in Example 3, it is easy to see that $h^\beta$ is a potential in $B(1)$. As in (8), we have

$$\mathcal{M}_p(h^\beta; r)^p \geq C(n, p)(1 - r)^{\beta p} \int_0^{1 - r} t^{n-2} \left\{ (1 - r)^2 + t^2 \right\}^{-np/2} \, dt$$

$$= C(n, p)(1 - r)^{(n - 1)(1 - 1/p)} \int_0^{\pi/4} \sin^{n-2} \phi \cos^{n(p\beta - 1)} \phi \, d\phi,$$

and so

$$(1 - r)^{(n - 1)(1 - 1/p) - \varepsilon} \mathcal{M}_p(h^\beta; r) \geq C(n, p)(1 - r)^{-\varepsilon/n} \to +\infty \quad (r \to 1^-)$$

as claimed.
REFERENCES


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