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GROWTH PROPERTIES OF pTH MEANS OF POTENTIALS IN THE UNIT BALL

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ABSTRACT. Let v be a potential in the unit ball of $\mathbf{R}^n$, and $\mathscr{M}_p(v;r)$ be its pth order mean over the sphere of radius r centred at the origin. It is shown that, as $r \to 1-$, the function $f(r) = (1-r)^{(n-1)(1-1/p)}\mathscr{M}_p(v;r)$ has limit 0 when $1 \le p < (n-1)/(n-2)$, and has lower limit 0 when $n \ge 3$ and $(n-1)/(n-2) \le p < (n-1)/(n-3)$. This extends a result of Stoll, who showed that, when $n = 2$ and $p = +\infty$, $\liminf_{r\to 1-} f(r) = 0$. Examples are given to show that the theorems presented are best possible.

1. Introduction and results.

THEOREM A. *If v is a (Green) potential in the unit disc, then*

$$\liminf_{r\to 1-}(1-r)\sup_{0\le\theta<2\pi} v(re^{i\theta}) = 0.$$

In the special case where $v = -\log|B|$ and B is a convergent Blaschke product in the unit disc, Theorem A is due to Heins [**1**] (see also [**3**]). The result for general potentials was only recently established by Stoll [**4**] (see also [**2**]). It does not have an obvious analogue for potentials in the unit ball of $\mathbf{R}^n$ $(n \ge 3)$, for such functions can be valued identically $+\infty$ on a given radius.

We denote an arbitrary point of Euclidean space $\mathbf{R}^n$ $(n \ge 2)$ by $X = (x_1, \ldots, x_n)$ and put $|X| = (x_1^2 + \cdots + x_n^2)^{1/2}$. Let

$$B(r) = \{X \in \mathbf{R}^n : |X| < r\}, \qquad S(r) = \{X \in \mathbf{R}^n : |X| = r\},$$
$$A(r_1, r_2) = \{X \in \mathbf{R}^n : r_1 < |X| < r_2\},$$

and let $\hat{\sigma}$ denote normalized surface area measure on $S(r)$. If f is a nonnegative measurable function on $S(r)$, define

$$\mathscr{M}_p(f;r) = \left\{\int_{S(r)} f^p\, d\hat{\sigma}\right\}^{1/p} \qquad (p > 0)$$

and

$$\mathscr{M}_\infty(f;r) = \sup\{f(X) : X \in S(r)\}.$$

As will be seen below, Theorem A has the following analogue in higher dimensions:

$$\liminf_{r\to 1-}(1-r)\mathscr{M}_{(n-1)/(n-2)}(v;r) = 0$$

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for any potential v in $B(1)$. However, recalling the well-known property that $\mathscr{M}_1(v;r) \to 0$ as $r \to 1-$, it is natural to consider the limiting behaviour of $\mathscr{M}_p(v;\cdot)$ for other values of p. This leads to the following results.

THEOREM 1. *If v is a potential in $B(1) \subset \mathbf{R}^n$ $(n \geq 2)$, and $1 < p < (n-1)/(n-2)$, then*

$$(1) \qquad (1-r)^{(n-1)(1-1/p)}\mathscr{M}_p(v;r) \to 0 \qquad (r \to 1-).$$

THEOREM 2. *If v is a potential in $B(1) \subset \mathbf{R}^n$ $(n \geq 3)$, and $(n-1)/(n-2) \leq p < (n-1)/(n-3)$, then*

$$(2) \qquad \liminf_{r \to 1-}(1-r)^{(n-1)(1-1/p)}\mathscr{M}_p(v;r) = 0.$$

In later sections of the paper we prove Theorems 1 and 2 using ideas from another paper by Stoll [**5**]. First we mention some examples to show that these results are best possible.

EXAMPLE 1. *If $0 < p < 1$ and v is a potential in $B(1) \subset \mathbf{R}^n$ $(n \geq 2)$, it is a simple consequence of Jensen's inequality that $\mathscr{M}_p(v;r) \to 0$ as $r \to 1-$. However, neither* (1) *nor* (2) *hold. To see this, let $\beta = \min\{1, (n-1)(p^{-1}-1)\}$. Since $(1-|X|)$ is a potential in $B(1)$ and $\beta \in (0,1]$, it follows easily that $v_0(X) = (1-|X|)^\beta$ is also a potential in $B(1)$, but*

$$(1-r)^{(n-1)(1-1/p)}\mathscr{M}_p(v_0;r) \geq 1 \qquad (0 < r < 1).$$

EXAMPLE 2. *If $n \geq 3$, there is a potential v in $B(1)$ such that $\mathscr{M}_p(v;r) \equiv +\infty$ for all $p \geq (n-1)/(n-3)$. In fact, if $n = 3$, let $u(X) = -\log(x_2^2 + x_3^2)$, let h be the greatest harmonic minorant of u in $B(1)$, and define $v = u - h$. Then v is a potential and it is easy to see that $\mathscr{M}_\infty(v;r) \equiv +\infty$. If $n \geq 4$, let $u(X) = (x_2^2 + \cdots + x_n^2)^{(3-n)/2}$, and define the corresponding potential v as before. Straightforward estimates show that $\mathscr{M}_p(v;r) \equiv +\infty$ for $p \geq (n-1)/(n-3)$.*

It is also easy to see that, if $(n-1)/(n-2) \leq p < (n-1)/(n-3)$ and the measure associated with a potential v in $B(1)$ comprises point masses arbitrarily close to $S(1)$, then

$$\limsup_{r \to 1-}(1-r)^{(n-1)(1-1/p)}\mathscr{M}_p(v;r) = +\infty.$$

EXAMPLE 3. *Theorems* 1 *and* 2 *fail if we replace "potential" by "positive superharmonic function". To see this, let h be the positive harmonic function given by*

$$h(X) = (1-|X|^2)|X - (1,0,\ldots,0)|^{-n} \qquad (X \in B(1)),$$

and let $p \geq 1$. Then $(1-r)^{(n-1)(1-1/p)}\mathscr{M}_p(h;r)$ has a finite positive limit as $r \to 1-$. However, Theorems 1 *and* 2 *do hold for positive superharmonic functions which do not majorize any positive multiple of a Poisson kernel. Details are given in* §§5.1, 5.2.

EXAMPLE 4. *If $\varepsilon > 0$ and $p \geq 1$, then there is a potential v in $B(1)$ such that*

$$(1-r)^{(n-1)(1-1/p)-\varepsilon}\mathscr{M}_p(v;r) \to +\infty \qquad (r \to 1-).$$

Details may be found in §5.3.

2. Two preliminary lemmas.

2.1. Let O denote the origin of $\mathbf{R}^n$ and X^* be the image of a point X under inversion of centre O and radius 1. We use $G(\cdot,\cdot)$ to denote the Green kernel of $B(1)$ so that, in the case $n = 2$,

$$G(X,Y) = \begin{cases} -\log|Y-X| + \log(|X|\cdot|Y-X^*|) & (X \neq O), \\ -\log|Y| & (X = O), \end{cases}$$

and in the case $n \geq 3$,

$$G(X,Y) = \begin{cases} |Y-X|^{2-n} - (|X|\cdot|Y-X^*|)^{2-n} & (X \neq O), \\ |Y|^{2-n} - 1 & (X = O). \end{cases}$$

We recall that there is a one-to-one correspondence between potentials v on $B(1)$ and measures μ on $B(1)$ which satisfy

$$\int_{B(1)} (1-|Y|)\, d\mu(Y) < +\infty. \tag{3}$$

For the remainder of §2, we will assume that either $n = 2$ and $p > 1$, or $n \geq 3$ and $1 < p < (n-1)/(n-3)$. It will be convenient to let $\alpha = (n-1)(1-p^{-1})$, which clearly lies in the interval $(0,1)$ when $n = 2$, and $(0,2)$ when $n \geq 3$. Also, $C(a,b,c,\dots)$ will denote a positive constant depending at most on $a,b,c,\dots$, not necessarily the same on any two occurrences.

2.2. Let $E(r) = A((5r-1)/4, (3r+1)/4)$ so that, if $Y \in B(1)\backslash E(r)$ and $X \in S(r)$, then $|Y-X| \geq ||Y|-r| > (1-r)/4$.

LEMMA 1. *If p, α are as above and μ is a measure on $B(1)$ satisfying* (3), *then*

$$(1-r)^\alpha \int_{B(1)\backslash E(r)} \mathscr{M}_p(G(\cdot,Y);r)\, d\mu(Y) \to 0 \qquad (r \to 1-).$$

To see this, let $|X| = r$ and $Y \in B(1)\backslash E(r)$. If $n \geq 3$, then

$$\begin{aligned} G(X,Y) &= \{|Y-X|\cdot|rY - r^{-1}X|\}^{2-n}\{|rY-r^{-1}X|^{n-2} - |Y-X|^{n-2}\} \\ &\leq 2^{-1}(n-2)|Y-X|^{-n}\{|rY-r^{-1}X|^2 - |Y-X|^2\} \\ &\leq 2^{2\alpha-1}(n-2)(1-r)^{-\alpha}|Y-X|^{\alpha-n}\{|rY-r^{-1}X|^2 - |Y-X|^2\}. \end{aligned}$$

If $n = 2$, then

$$\begin{aligned} G(X,Y) &= \log\{|rY-r^{-1}X|/|Y-X|\} \\ &\leq |Y-X|^{-1}\{|rY-r^{-1}X| - |Y-X|\} \\ &\leq 2^{2\alpha-1}(1-r)^{-\alpha}|Y-X|^{\alpha-2}\{|rY-r^{-1}X|^2 - |Y-X|^2\}. \end{aligned}$$

Thus, letting $\rho = |Y|$ we have, for any $n \geq 2$,

$$\begin{aligned} &\int_{S(r)} [G(X,Y)]^p\, d\hat{\sigma}(X) \leq C(n,p)(1-r)^{-\alpha p} \\ &\quad\times \int_0^\pi \sin^{n-2}\theta\{(\rho - r\cos\theta)^2 + (r\sin\theta)^2\}^{(1-n-p)/2}\{(1-r^2)(1-\rho^2)\}^p\, d\theta \\ &= C(n,p)(1-r)^{-\alpha p}\{(1-r^2)(1-\rho^2)\}^p \\ &\quad\times \int_0^\infty t^{n-2}(1+t^2)^{(p+1-n)/2}\{(\rho-r)^2 + (\rho+r)^2t^2\}^{(1-n-p)/2}\, dt, \end{aligned}$$

using the substitution $t = \tan(\theta/2)$. Hence, if $\kappa = |\rho - r|/(\rho + r)$ and $r > \frac{1}{2}$,

$$\begin{aligned}
&\{(1-r)^\alpha \mathscr{M}_p(G(\cdot, Y); r)\}^p \\
&\quad \le C(n,p)\{(1-r)(1-\rho)\}^p \int_0^\infty t^{n-2}(1+t^2)^{(p+1-n)/2}(\kappa^2+t^2)^{(1-n-p)/2}\,dt \\
&\quad = C(n,p)\{(1-r)(1-\rho)/\kappa\}^p \\
&\quad\quad \times \int_0^\infty x^{n-2}(1+\kappa^2x^2)^{(p+1-n)/2}(1+x^2)^{(1-n-p)/2}\,dx \\
&\quad \le C(n,p)\{(1-r)(1-\rho)/\kappa\}^p \int_0^\infty x^{n-2}(1+x^2)^{[1-n-p+(p+1-n)^+]/2}\,dx \\
&\quad \le C(n,p)\{(1-r)(1-\rho)/|\rho - r|\}^p
\end{aligned}$$

by means of the substitution $t = \kappa x$. Thus

$$(1-r)^\alpha \mathscr{M}_p(G(\cdot, Y); r) \le C(n,p)(1-|Y|) \qquad (Y \in B(1)\backslash E(r)). \tag{4}$$

Now let $\varepsilon > 0$ and choose $R \in (0,1)$ large enough to ensure that

$$\int_{A(R,1)} (1-|Y|)\,d\mu(Y) < \varepsilon.$$

Simple estimates yield

$$G(X,Y) \le C(n,R)(1-|X|) \qquad (|Y| \le R;\ (1+R)/2 < |X| < 1).$$

Hence, if $r \in ((1+R)/2, 1)$, it follows from (4) that

$$\begin{aligned}
(1-r)^\alpha &\int_{B(1)\backslash E(r)} \mathscr{M}_p(G(\cdot, Y); r)\,d\mu(Y) \\
&\le C(n,R)(1-r)^{\alpha+1}\mu(\{|Y| \le R\}) + C(n,p)\varepsilon \\
&\to C(n,p)\varepsilon \qquad (r \to 1-).
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the lemma is proved.

2.3.

LEMMA 2. *Let p, α be as above, let $r > \frac{1}{2}$ and $Y \in E(r)$. Then*

$$(1-|Y|)^{-1}(1-r)^\alpha \mathscr{M}_p(G(\cdot, Y); r) \le \begin{cases} C(n,p) & (0 < \alpha < 1), \\ C(n,p)\log[(1-r)/||Y|-r||] & (\alpha = 1), \\ C(n,p)\,[(1-r)/||Y|-r||]^{\alpha-1} & (1 < \alpha < 2). \end{cases}$$

To prove the lemma, let $r > \frac{1}{2}$, $X \in S(r)$, $Y \in E(r)$ and $\rho = |Y|$. Then

$$|rY - r^{-1}X| \le |rY - Y| + |Y - X| + |X - r^{-1}X| < 2(1-r) + |Y - X|$$

and

$$|\rho - r| < (1-r)/4 < (1-\rho)/3.$$

We deal first with the case $n \ge 3$, where

$$\begin{aligned}
G(X,Y) &\le (n-2)|Y-X|^{2-n}\{1 - |Y-X|/|rY - r^{-1}X|\} \\
&\le 2(n-2)(1-r)|Y-X|^{2-n}/\{2(1-r) + |Y-X|\}.
\end{aligned}$$

Hence, as in the proof of Lemma 1,

$$\begin{aligned}
\int_{S(r)} &[G(X,Y)]^p \, d\hat{\sigma}(X) \\
&\le C(n,p)(1-r)^p \int_0^\pi \sin^{n-2}\theta\{(\rho - r\cos\theta)^2 + (r\sin\theta)^2\}^{p-np/2} \\
&\qquad \times \{2(1-r) + \left[(\rho - r\cos\theta)^2 + (r\sin\theta)^2\right]^{1/2}\}^{-p} \, d\theta \\
&\le C(n,p)(1-r)^p \int_0^\infty t^{n-2}(1+t^2)^{(n-1)(p-2)/2}(\kappa^2+t^2)^{p-np/2} \\
&\qquad \times \{(1-r)(1+t^2)^{1/2} + (\kappa^2+t^2)^{1/2}\}^{-p} \, dt \\
&\le C(n,p)(1-r)^p \int_0^\infty t^{n-2}(1+t^2)^{(n-1)(p-2)/2} \\
&\qquad \times (\kappa^2+t^2)^{p-np/2}\{(1-r)+t\}^{-p} \, dt
\end{aligned} \tag{5}$$

where $\kappa = |\rho - r|/(\rho + r)$. We split up the integral in (5) into the components J_1, J_2, J_3, J_4 corresponding to the intervals $[0,\kappa], [\kappa, 1-r], [1-r, 1], [1, \infty)$. Then

$$\begin{aligned}
J_1 &\le C(n,p)(1-r)^{-p} \int_0^\kappa t^{n-2}(\kappa^2+t^2)^{p-np/2} \, dt \\
&= C(n,p)(1-r)^{-p}\kappa^{p(1-\alpha)} \int_0^{\pi/4} \sin^{n-2}\phi \cos^{pn-2p-n}\phi \, d\phi
\end{aligned}$$

using the substitution $t = \kappa \tan\phi$. Next

$$\begin{aligned}
J_2 &\le C(n,p)(1-r)^{-p} \int_\kappa^{1-r} t^{(n-2)(1-p)} \, dt \\
&= C(n,p)(1-r)^{-p} \int_\kappa^{1-r} t^{-p(\alpha-1)-1} \, dt \\
&\le \begin{cases} C(n,p)(1-r)^{-p\alpha} & (0 < \alpha < 1), \\ C(n,p)(1-r)^{-p}\log\{(1-r)/\kappa\} & (\alpha = 1), \\ C(n,p)(1-r)^{-p}\kappa^{p(1-\alpha)} & (1 < \alpha < 2), \end{cases} \\
J_3 &\le C(n,p) \int_{1-r}^1 t^{(n-2)(1-p)}(1-r+t)^{-p} \, dt \\
&\le C(n,p) \int_{1-r}^1 (1-r+t)^{p+n-np-2} \, dt \\
&\le C(n,p)(1-r)^{-\alpha p}, \\
J_4 &\le C(n,p) \int_1^\infty t^{-n} \, dt = C(n,p).
\end{aligned}$$

Hence, from (5),

$$\int_{S(r)} [G(X,Y)]^p \, d\hat{\sigma}(X) \le \begin{cases} C(n,p)(1-r)^{p(1-\alpha)} & (0 < \alpha < 1), \\ C(n,p)\log\{(1-r)/|\rho - r|\} & (\alpha = 1), \\ C(n,p)|\rho - r|^{p(1-\alpha)} & (1 < \alpha < 2). \end{cases}$$

The $n \ge 3$ case of the lemma now follows easily on taking pth roots and noting that $\log\{(1-r)/|\rho - r|\} > \log 4 > 1$.

The $n = 2$ case of the lemma is more straightforward, as here we have $0 < \alpha < 1$. Since

$$G(X,Y) \leq \log\{1 + 2(1-r)/|Y - X|\},$$

it follows as before that

$$\begin{aligned}
&\int_{S(r)} [G(X,Y)]^p \, d\hat{\sigma}(X) \\
&\quad \leq \pi^{-1} \int_0^{\pi} \left\{\log\left[1 + 2(1-r)\{(\rho - r\cos\theta)^2 + (r\sin\theta)^2\}^{-1/2}\right]\right\}^p d\theta \\
&\quad \leq 2\pi^{-1} \int_0^{\infty} (1+t^2)^{-1} \left\{\log\left[1 + 4(1-r)(1+t^2)^{1/2}(\kappa^2 + t^2)^{-1/2}\right]\right\}^p dt. \qquad (6)
\end{aligned}$$

We split up the integral in (6) into the components J_1, J_2, J_3 corresponding to the intervals $[0, 1-r], [1-r, 1], [1, \infty)$. Then

$$\begin{aligned}
J_1 &\leq \int_0^{1-r} \{\log[7(1-r)t^{-1}]\}^p \, dt = C(p)(1-r), \\
J_2 &\leq \int_{1-r}^{1} \{\log[1 + 6(1-r)t^{-1}]\}^p \, dt \\
&\leq 6^p (1-r)^p \int_{1-r}^{1} t^{-p} \, dt \leq C(p)(1-r), \\
J_3 &\leq \int_1^{\infty} t^{-2} \{\log[1 + 6(1-r)]\}^p \, dt \leq 6^p (1-r)^p,
\end{aligned}$$

and so from (6)

$$\int_{S(r)} [G(X,Y)]^p \, d\hat{\sigma}(X) \leq C(p)(1-r).$$

Hence

$$(1 - |Y|)^{-1}(1-r)^{\alpha} \mathscr{M}_p(G(\cdot, Y); r) \leq \tfrac{4}{3}(1-r)^{-1/p} \mathscr{M}_p(G(\cdot, Y); r) \leq C(p),$$

as required.

3. Proof of Theorem 1. Let $1 < p < (n-1)/(n-2)$, so that $0 < \alpha < 1$, and let μ be the measure corresponding to the potential v. By Minkowski's inequality,

$$\begin{aligned}
\mathscr{M}_p(v; r) &= \left\{\int_{S(r)} \left[\int_{B(1)} G(X,Y) \, d\mu(Y)\right]^p d\hat{\sigma}(X)\right\}^{1/p} \\
&\leq \int_{B(1)} \mathscr{M}_p(G(\cdot, Y); r) \, d\mu(Y).
\end{aligned}$$

Thus, from Lemmas 1 and 2,

$$(1-r)^{\alpha} \mathscr{M}_p(v; r) \leq o(1) + C(n,p) \int_{E(r)} (1 - |Y|) \, d\mu(Y) = o(1) \qquad (r \to 1-)$$

in the light of (3).

4. Proof of Theorem 2.

4.1. If μ is a measure on $B(1)$ satisfying (3), we define a finite measure μ^* on $B(1)$ by $d\mu^*(Y) = (1-|Y|)\,d\mu(Y)$. For any interval $I \subseteq [0,1)$, let $A(I) = \{X : |X| \in I\}$. The maximal function $M(d\mu^*)$ is defined by

$$M(d\mu^*)(t) = \sup_{I \ni t} \mu^*(A(I))/|I|,$$

where $|I|$ denotes the Lebesgue measure of I. Also, for each $k \in \mathbf{N}$, we define the interval $I_k = [1-2^{-k}, 1-2^{-k-1})$, and denote by μ_k^* the restriction of μ^* to $A(I_k)$. Well-known estimates for the maximal function yield the following lemma (cf. [**5**, p. 453]).

LEMMA A. *For each $k \in \mathbf{N}$, there exists $r_k \in (1-2^{2-k}/5, 1-2^{1-k}/3)$ such that*

$$M(d\mu_k^*)(r_k) \le 2^{k+4}\mu^*(A(I_k)).$$

4.2. We now prove Theorem 2. Let $n \ge 3$ and $(n-1)/(n-2) \le p < (n-1)/(n-3)$, so that $1 \le \alpha < 2$, and let μ be the measure corresponding to the potential v. We give below the argument for $1 < \alpha < 2$, the case $\alpha = 1$ being similar.

As in §3,

$$(1-r)^\alpha \mathscr{M}_p(v;r) \le o(1) + C(n,p)\int_{E(r)} \left[\frac{1-r}{||Y|-r|}\right]^{\alpha-1} d\mu^*(Y) \tag{7}$$

as $r \to 1-$. Now let (r_k) be a sequence as in Lemma A. Fixing k and approximating $[(1-r_k)/|x-r_k|]^{\alpha-1}$ by a monotone increasing sequence of step functions of the form

$$f(x) = \sum_{j=1}^{N} a_j \chi_{I_j}(x),$$

where each a_j is nonnegative and each interval I_j is symmetric about r_k it follows that

$$\begin{aligned}\int_{E(r_k)} \left[\frac{1-r_k}{||Y|-r_k|}\right]^{\alpha-1} d\mu^*(Y) &\le 2M(d\mu_k^*)(r_k)(1-r_k)^{\alpha-1}\int_0^{(1-r_k)/4} x^{1-\alpha}\,dx \\ &= C(n,p)(1-r_k)M(d\mu_k^*)(r_k).\end{aligned}$$

Since, from Lemma A,

$$(1-r_k)M(d\mu_k^*)(r_k) \le 2^4\mu^*(A(I_k)),$$

it now follows from (7) that

$$(1-r_k)^\alpha \mathscr{M}_p(v;r_k) \le o(1) + C(n,p)\mu^*(A(I_k)) = o(1) \qquad (k \to \infty),$$

as required.

5. Examples 3 and 4.

5.1. Let h be as in Example 3 and $p \ge 1$. Then

$$\begin{aligned}&\int_{S(r)} [h(X)]^p\,d\hat{\sigma}(X) \\ &= C(n)(1-r^2)^p \int_0^\pi \sin^{n-2}\theta\{(1-r\cos\theta)^2 + (r\sin\theta)^2\}^{-np/2}\,d\theta \\ &= C(n)(1-r^2)^p \int_0^\infty t^{n-2}(1+t^2)^{1-n+np/2}\{(1-r)^2 + (1+r)^2t^2\}^{-np/2}\,dt.\end{aligned} \tag{8}$$

Splitting up the integral in (8) into the components J_1, J_2 corresponding to the intervals $[0,1], [1,\infty)$, it is clear from dominated convergence that $J_2(r)$ has a finite limit as $r \to 1-$. Also

$$(1-r)^{np+1-n} J_1(r) = (1+r)^{1-n} \int_0^{\tan^{-1}\{(1+r)/(1-r)\}} \sin^{n-2}\phi \cos^{n(p-1)}\phi \times \{1 + (1-r)^2(1+r)^{-2}\tan^2\phi\}^{1-n+np/2}\, d\phi$$

has a finite positive limit as $r \to 1-$, so

$$(1-r)^{(n-1)(1-1/p)} \mathscr{M}_p(h; r) \to C(n,p) \qquad (r \to 1-). \tag{9}$$

5.2. By Minkowski's inequality it is sufficient to show that

$$(1-r)^{(n-1)(1-1/p)} \mathscr{M}_p(h; r) \to 0 \qquad (r \to 1-) \tag{10}$$

for any positive harmonic function h in $B(1)$ which does not majorize any positive multiple of a Poisson kernel. For simplicity we show this when $n = 2$; the higher dimensional argument requires minor modification.

Let μ be the measure on $[0, 2\pi)$ associated with h in the Poisson integral representation. From §5.1 it is straightforward to deduce that

$$\limsup_{r \to 1-} (1-r)^{(p-1)} \int_\alpha^\beta \{h(re^{i\theta})\}^p\, d\theta \leq C(p)\{\mu([\alpha, \beta])\}^p$$

for $0 \leq \alpha < \beta < 2\pi$. Let k be a positive integer. No singleton has positive measure, so we can let $0 = \gamma_0 < \gamma_1 < \cdots < \gamma_k = 2\pi$ be such that

$$\mu([\gamma_{i-1}, \gamma_i]) = \mu([0, 2\pi))/k$$

for each $i \in \{1, \ldots, k\}$. Then

$$\limsup_{r \to 1-} (1-r)^{(p-1)} \int_0^{2\pi} \{h(re^{i\theta})\}^p\, d\theta \leq C(p)\{\mu([0, 2\pi))\}^p k^{1-p}.$$

Since k can be arbitrarily large, (10) holds for any $p > 1$.

5.3. To establish Example 4, let $\varepsilon > 0$ and $p \geq 1$, and choose $\beta \in (1 - \varepsilon/n, 1)$. If h is as in Example 3, it is easy to see that h^β is a potential in $B(1)$. As in (8), we have

$$\begin{aligned}\{\mathscr{M}_p(h^\beta; r)\}^p &\geq C(n,p)(1-r)^{\beta p} \int_0^{1-r} t^{n-2}\{(1-r)^2 + t^2\}^{-np\beta/2}\, dt \\ &= C(n,p)(1-r)^{(n-1)(1-p\beta)} \int_0^{\pi/4} \sin^{n-2}\phi \cos^{n(p\beta-1)}\phi\, d\phi,\end{aligned}$$

and so

$$(1-r)^{(n-1)(1-1/p)-\varepsilon} \mathscr{M}_p(h^\beta; r) \geq C(n,p)(1-r)^{-\varepsilon/n} \to +\infty \qquad (r \to 1-)$$

as claimed.

REFERENCES

1. M. Heins, *The minimum modulus of a bounded analytic functions*, Duke Math. J. **14** (1947), 179–215.
2. W. Nestlerode and M. Stoll, *Radial limits of n-subharmonic functions in the polydisc*, Trans. Amer. Math. Soc. **279** (1983), 691–703.
3. J. H. Shapiro and A. L. Shields, *Unusual topological properties of the Nevanlinna class*, Amer. J. Math. **97** (1976), 915–936.
4. M. Stoll, *Boundary limits of subharmonic functions in the disc*, Proc. Amer. Math. Soc. **93** (1985), 567–568.
5. ——, *Boundary limits of Green potentials in the unit disc*, Arch. Math. **44** (1985), 451–455.

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