GROWTH PROPERTIES OF pTH MEANS OF POTENTIALS IN THE UNIT BALL

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(Communicated by Irwin Kra)

ABSTRACT. Let \( v \) be a potential in the unit ball of \( \mathbb{R}^n \), and \( \mathcal{M}_p(v;r) \) be its pth order mean over the sphere of radius \( r \) centred at the origin. It is shown that, as \( r \to 1^- \), the function \( f(r) = (1 - r)\left(\frac{n-1}{(n-2)}\right)\mathcal{M}_p(v;r) \) has limit 0 when \( 1 < p < \frac{n-1}{(n-2)} \), and has lower limit 0 when \( n > 3 \) and \( \frac{n-1}{(n-2)} < p < \frac{n-1}{(n-3)} \). This extends a result of Stoll, who showed that, when \( n = 2 \) and \( p = +\infty \), \( \liminf_{r \to 1^-} f(r) = 0 \). Examples are given to show that the theorems presented are best possible.

1. Introduction and results.

**THEOREM A.** If \( v \) is a (Green) potential in the unit disc, then
\[
\liminf_{r \to 1^-} \frac{1}{(1 - r)^{n-1}} \sup_{0 \leq \theta < 2\pi} v(r e^{i\theta}) = 0.
\]

In the special case where \( v = -\log |B| \) and \( B \) is a convergent Blaschke product in the unit disc, Theorem A is due to Heins [1] (see also [3]). The result for general potentials was only recently established by Stoll [4] (see also [2]). It does not have an obvious analogue for potentials in the unit ball of \( \mathbb{R}^n \) (\( n \geq 3 \)), for such functions can be valued identically \( +\infty \) on a given radius.

We denote an arbitrary point of Euclidean space \( \mathbb{R}^n \) (\( n \geq 2 \)) by \( X = (x_1, \ldots, x_n) \) and put \( |X| = (x_1^2 + \cdots + x_n^2)^{1/2} \). Let
\[
B(r) = \{X \in \mathbb{R}^n : |X| < r\}, \quad S(r) = \{X \in \mathbb{R}^n : |X| = r\},
\]
\[
A(r_1, r_2) = \{X \in \mathbb{R}^n : r_1 < |X| < r_2\},
\]
and let \( \sigma \) denote normalized surface area measure on \( S(r) \). If \( f \) is a nonnegative measurable function on \( S(r) \), define
\[
\mathcal{M}_p(f;r) = \left\{ \int_{S(r)} f^p \sigma \right\}^{1/p} \quad (p > 0)
\]
and
\[
\mathcal{M}_\infty(f;r) = \sup\{f(X) : X \in S(r)\}.
\]

As will be seen below, Theorem A has the following analogue in higher dimensions:
\[
\liminf_{r \to 1^-} (1 - r)^{\frac{n-1}{(n-2)}} \mathcal{M}_p((n-1)/(n-2)(v;r)) = 0
\]
for any potential $v$ in $B(1)$. However, recalling the well-known property that $\mathcal{M}_1(v;r) \to 0$ as $r \to 1-$, it is natural to consider the limiting behaviour of $\mathcal{M}_p(v;\cdot)$ for other values of $p$. This leads to the following results.

**Theorem 1.** If $v$ is a potential in $B(1) \subset \mathbb{R}^n$ ($n \geq 2$), and $1 < p < (n-1)/(n-2)$, then
\[
(1 - r)^{(n-1)(1-1/p)} \mathcal{M}_p(v;r) \to 0 \quad (r \to 1-).
\]

**Theorem 2.** If $v$ is a potential in $B(1) \subset \mathbb{R}^n$ ($n \geq 3$), and $(n-1)/(n-2) < p < (n-1)/(n-3)$, then
\[
\liminf_{r \to 1-} (1 - r)^{(n-1)(1-1/p)} \mathcal{M}_p(v;r) = 0.
\]

In later sections of the paper we prove Theorems 1 and 2 using ideas from another paper by Stoll [5]. First we mention some examples to show that these results are best possible.

**Example 1.** If $0 < p < 1$ and $v$ is a potential in $B(1) \subset \mathbb{R}^n$ ($n \geq 2$), it is a simple consequence of Jensen’s inequality that $\mathcal{M}_p(v;r) \to 0$ as $r \to 1-$. However, neither (1) nor (2) hold. To see this, let $\beta = \min\{1, (n-1)(p-1-1)\}$. Since $(1-|X|)$ is a potential in $B(1)$ and $\beta \in (0,1]$, it follows easily that $v_0(X) = (1 - |X|)^\beta$ is also a potential in $B(1)$, but
\[
(1 - r)^{(n-1)(1-1/p)} \mathcal{M}_p(v_0;r) \geq 1 \quad (0 < r < 1).
\]

**Example 2.** If $n \geq 3$, there is a potential $v$ in $B(1)$ such that $\mathcal{M}_{\infty}(v;r) \equiv +\infty$ for all $p \geq (n-1)/(n-3)$. In fact, if $n = 3$, let $u(X) = -\log(x_2^2 + x_3^2)$, let $h$ be the greatest harmonic minorant of $u$ in $B(1)$, and define $v = u - h$. Then $v$ is a potential and it is easy to see that $\mathcal{M}_{\infty}(v;r) \equiv +\infty$. If $n \geq 4$, let $u(X) = (x_2^2 + \cdots + x_n^2)^{(3-n)/2}$, and define the corresponding potential $v$ as before. Straightforward estimates show that $\mathcal{M}_p(v;r) \equiv +\infty$ for $p \geq (n-1)/(n-3)$.

It is also easy to see that, if $(n-1)/(n-2) \leq p < (n-1)/(n-3)$ and the measure associated with a potential $v$ in $B(1)$ comprises point masses arbitrarily close to $S(1)$, then
\[
\limsup_{r \to 1-} (1 - r)^{(n-1)(1-1/p)} \mathcal{M}_p(v;r) = +\infty.
\]

**Example 3.** Theorems 1 and 2 fail if we replace “potential” by “positive superharmonic function”. To see this, let $h$ be the positive harmonic function given by
\[
h(X) = (1 - |X|^2)|X - (1,0,\ldots,0)|^{-n} \quad (X \in B(1)),
\]
and let $p \geq 1$. Then $(1 - r)^{(n-1)(1-1/p)} \mathcal{M}_p(h;r)$ has a finite positive limit as $r \to 1-$. However, Theorems 1 and 2 do hold for positive superharmonic functions which do not majorize any positive multiple of a Poisson kernel. Details are given in §§5.1, 5.2.

**Example 4.** If $\epsilon > 0$ and $p \geq 1$, then there is a potential $v$ in $B(1)$ such that
\[
(1 - r)^{(n-1)(1-1/p)} - \epsilon \mathcal{M}_p(v;r) \to +\infty \quad (r \to 1-).
\]

Details may be found in §5.3.
2. Two preliminary lemmas.

2.1. Let $O$ denote the origin of $\mathbb{R}^n$ and $X^*$ be the image of a point $X$ under inversion of centre $O$ and radius 1. We use $G(\cdot, \cdot)$ to denote the Green kernel of $B(1)$ so that, in the case $n = 2$,

$$G(X, Y) = \begin{cases} -\log |Y - X| + \log(|X| \cdot |Y - X^*|) & (X \neq O), \\ -\log |Y| & (X = O), \end{cases}$$

and in the case $n \geq 3$,

$$G(X, Y) = \begin{cases} |Y - X|^{2-n} - (|X| \cdot |Y - X^*|)^{2-n} & (X \neq O), \\ |Y|^{2-n} - 1 & (X = O). \end{cases}$$

We recall that there is a one-to-one correspondence between potentials $v$ on $B(1)$ and measures $\mu$ on $B(1)$ which satisfy

$$(3) \quad \int_{B(1)} (1 - |Y|) d\mu(Y) < +\infty.$$ 

For the remainder of §2, we will assume that either $n = 2$ and $p > 1$, or $n \geq 3$ and $1 < p < (n-1)/(n-3)$. It will be convenient to let $\alpha = (n-1)(1-p^{-1})$, which clearly lies in the interval $(0,1)$ when $n = 2$, and $(0,2)$ when $n \geq 3$. Also, $C(a,b,c,\ldots)$ will denote a positive constant depending at most on $a, b, c, \ldots$, not necessarily the same on any two occurrences.

2.2. Let $E(r) = A((5r-1)/4, (3r+1)/4)$ so that, if $Y \in B(1) \setminus E(r)$ and $X \in S(r)$, then $|Y - X| \geq ||Y - r|| > (1 - r)/4$.

**Lemma 1.** If $p, \alpha$ are as above and $\mu$ is a measure on $B(1)$ satisfying (3), then

$$(1 - r)\alpha \int_{B(1) \setminus E(r)} \mathcal{F}(\cdot, Y; r) d\mu(Y) \to 0 \quad (r \to 1-).$$

To see this, let $|X| = r$ and $Y \in B(1) \setminus E(r)$. If $n \geq 3$, then

$$G(X, Y) = \left(|Y - X| \cdot |rY - r^{-1}X|\right)^{2-n} \left(|rY - r^{-1}X|^{n-2} - |Y - X|^{n-2}\right)$$

$$\leq 2^{-1}(n-2)|Y - X|^{-n} \left(|rY - r^{-1}X|^2 - |Y - X|^2\right)$$

$$\leq 2^{2\alpha - 1}(n-2)(1-r)^{-\alpha} |Y - X|^{\alpha - n} \left(|rY - r^{-1}X|^2 - |Y - X|^2\right).$$

If $n = 2$, then

$$G(X, Y) = \log\{|rY - r^{-1}X|/|Y - X|\}$$

$$\leq |Y - X|^{-1} \left(|rY - r^{-1}X| - |Y - X|\right)$$

$$\leq 2^{2\alpha - 1}(1-r)^{-\alpha} |Y - X|^{-2} \left(|rY - r^{-1}X|^2 - |Y - X|^2\right).$$

Thus, letting $\rho = |Y|$ we have, for any $n \geq 2$,

$$\int_{S(r)} [G(X, Y)]^p d\sigma(X) \leq C(n,p)(1-r)^{-\alpha p}$$

$$\times \int_0^\pi \sin^{n-2} \theta \left((\rho - r \cos \theta)^2 + (r \sin \theta)^2\right)^{(1-n-p)/2} \left((1-r^2)(1-\rho^2)\right)^p d\theta$$

$$= C(n,p)(1-r)^{-\alpha p} \left((1-r^2)(1-\rho^2)\right)^p$$

$$\times \int_0^\infty t^{n-2} (1 + t^2)^{(p+1-n)/2} \left((\rho - r)^2 + (\rho + r)^2 t^2\right)^{(1-n-p)/2} dt,$$
using the substitution \( t = \tan(\theta/2) \). Hence, if \( \kappa = |\rho - r|/(\rho + r) \) and \( r > \frac{1}{2} \),
\[
(1 - r)^\alpha \mathcal{M}_p(G(\cdot, Y); r)^p \leq C(n, p) \{(1 - r)(1 - \rho)\}^p \int_0^\infty t^{n-2}(1 + t^2)^{(p+1-n)/2}(\kappa^2 + t^2)^{(1-n-p)/2}\,dt
\]
\[
= C(n, p) \{(1 - r)(1 - \rho)/\kappa\}^p \times \int_0^\infty x^{n-2}(1 + \kappa^2 x^2)^{(p+1-n)/(1 + x^2)(1-n-p)/2}\,dx
\]
\[
\leq C(n, p) \{(1 - r)(1 - \rho)/\kappa\}^p \int_0^\infty x^{n-2}(1 + x^2)^{(1-n-p+(p+1-n)/2)}\,dx
\]
\[
\leq C(n, p) \{(1 - r)(1 - \rho)/|\rho - r|\}^p
\]
by means of the substitution \( t = \rho x \). Thus
\[
(1 - r)^\alpha \mathcal{M}_p(G(\cdot, Y); r) \leq C(n, p)(1 - |Y|) \quad (Y \in B(1) \setminus E(r)).
\]

Now let \( \varepsilon > 0 \) and choose \( R \in (0, 1) \) large enough to ensure that
\[
\int_{A(R, 1)} (1 - |Y|)\,d\mu(Y) < \varepsilon.
\]
Simple estimates yield
\[
G(X, Y) \leq C(n, R)(1 - |X|) \quad (|Y| \leq R; (1 + R)/2 < |X| < 1).
\]
Hence, if \( r \in ((1 + R)/2, 1) \), it follows from (4) that
\[
(1 - r)^\alpha \int_{B(1) \setminus E(r)} \mathcal{M}_p(G(\cdot, Y); r)\,d\mu(Y)
\]
\[
\leq C(n, R)(1 - r)^{\alpha+1}\mu(|Y| \leq R) + C(n, p)\varepsilon
\]
\[
\rightarrow C(n, p)\varepsilon \quad (r \rightarrow 1-).
\]
Since \( \varepsilon > 0 \) was arbitrary, the lemma is proved.

2.3.

**Lemma 2.** Let \( p, \alpha \) be as above, let \( r > \frac{1}{2} \) and \( Y \in E(r) \). Then
\[
(1 - |Y|)^{-1}(1 - r)^\alpha \mathcal{M}_p(G(\cdot, Y); r) \leq \begin{cases} 
C(n, p) & (0 < \alpha < 1), \\
C(n, p) \log [(1 - r)/|Y| - r] & (\alpha = 1), \\
C(n, p) [(1 - r)/|Y| - r]^{\alpha-1} & (1 < \alpha < 2).
\end{cases}
\]

To prove the lemma, let \( r > \frac{1}{2}, X \in S(r), Y \in E(r) \) and \( \rho = |Y| \). Then
\[
|rY - r^{-1}X| \leq |rY - Y| + |Y - X| + |X - r^{-1}X| < 2(1 - r) + |Y - X|
\]
and
\[
|\rho - r| < (1 - r)/4 < (1 - \rho)/3.
\]
We deal first with the case \( n \geq 3 \), where
\[
G(X, Y) \leq (n - 2)|Y - X|^{2-n} \{1 - |Y - X|/rY - r^{-1}X\}
\]
\[
\leq 2(n - 2)(1 - r)|Y - X|^{2-n}/\{2(1 - r) + |Y - X|\}.
\]
Hence, as in the proof of Lemma 1,
\[ \int_{S(r)} [G(X,Y)]^p \, d\sigma(X) \leq C(n,p)(1-r)^p \int_0^{\pi} \sin^{n-2}\theta \{(\rho - r \cos\theta)^2 + (r \sin\theta)^2\}^{p-np/2} \times \{2(1-r) + [(\rho - r \cos\theta)^2 + (r \sin\theta)^2]^{1/2}\}^{-p} \, d\theta \]
\[ \leq C(n,p)(1-r)^p \int_0^{\infty} t^{n-2}(1 + t^2)^{(n-1)(p-2)/2}(\kappa^2 + t^2)^{p-np/2} \times \{(1-r)(1+t^2)^{1/2} + (\kappa^2 + t^2)^{1/2}\}^{-p} \, dt \]
\[ \leq C(n,p)(1-r)^p \int_0^{\infty} t^{n-2}(1 + t^2)^{(n-1)(p-2)/2} \times (\kappa^2 + t^2)^{p-np/2} \{(1-r) + t\}^{-p} \, dt \]
where \( \kappa = |\rho - r|/(\rho + r) \). We split up the integral in (5) into the components \( J_1, J_2, J_3, J_4 \) corresponding to the intervals \([0, \kappa], [\kappa, 1-r], [1-r, 1], [1, \infty)\). Then
\[ J_1 \leq C(n,p)(1-r)^{-p} \int_0^\kappa t^{n-2}(\kappa^2 + t^2)^{p-np/2} \, dt \]
\[ = C(n,p)(1-r)^{-p} \kappa^{p(1-\alpha)} \int_0^{\pi/4} \sin^{n-2} \phi \cos^{pn-2p-n} \phi \, d\phi \]
using the substitution \( t = \kappa \tan \phi \). Next
\[ J_2 \leq C(n,p)(1-r)^{-p} \int_\kappa^{1-r} t^{n-2}(1-t)^{-p} \, dt \]
\[ = C(n,p)(1-r)^{-p} \int_\kappa^{1-r} t^{-p(\alpha-1)-1} \, dt \]
\[ \leq \begin{cases} C(n,p)(1-r)^{-p\alpha} \quad (0 < \alpha < 1), \\
C(n,p)(1-r)^{-p} \log\{(1-r)/\kappa\} \quad (\alpha = 1), \\
C(n,p)(1-r)^{-p} \kappa^{p(1-\alpha)} \quad (1 < \alpha < 2), \end{cases} \]
\[ J_3 \leq C(n,p) \int_{1-r}^1 t^{n-2}(1-t)^{-p} (1-r+t)^{-p} \, dt \]
\[ \leq C(n,p) \int_{1-r}^1 (1-r+t)^{p+n-np-2} \, dt \]
\[ \leq C(n,p)(1-r)^{-\alpha p}, \]
\[ J_4 \leq C(n,p) \int_1^{\infty} t^{-n} \, dt = C(n,p). \]
Hence, from (5),
\[ \int_{S(r)} [G(X,Y)]^p \, d\sigma(X) \leq \begin{cases} C(n,p)(1-r)^{p(1-\alpha)} \quad (0 < \alpha < 1), \\
C(n,p) \log\{(1-r)/|\rho - r|\} \quad (\alpha = 1), \\
C(n,p)|\rho - r|^{p(1-\alpha)} \quad (1 < \alpha < 2). \end{cases} \]
The \( n \geq 3 \) case of the lemma now follows easily on taking \( p \)th roots and noting that \( \log\{(1-r)/|\rho - r|\} > \log 4 > 1 \).
The \( n = 2 \) case of the lemma is more straightforward, as here we have \( 0 < \alpha < 1 \). Since

\[ G(X, Y) \leq \log\{1 + 2(1 - r)/|Y - X|\}, \]

it follows as before that

\[
\int_{S(r)} [G(X, Y)]^p \, d\hat{\sigma}(X) \\
\leq \pi^{-1} \int_0^\pi \left\{ \log \left[ 1 + 2(1 - r)\{ (\rho - r \cos \theta)^2 + (r \sin \theta)^2 \}^{-1/2} \right] \right\}^p \, d\theta \\
\leq 2\pi^{-1} \int_0^\infty (1 + t^2)^{-1} \left\{ \log \left[ 1 + 4(1 - r)(1 + t^2)^{1/2}(\kappa^2 + t^2)^{-1/2} \right] \right\}^p \, dt.
\]

We split up the integral in (6) into the components \( J_1, J_2, J_3 \) corresponding to the intervals \([0, 1 - r], [1 - r, 1], [1, \infty)\). Then

\[
J_1 \leq \int_0^{1 - r} \{ \log \left[ 7(1 - r)t^{-1} \right] \}^p \, dt = C(p)(1 - r), \\
J_2 \leq \int_{1 - r}^1 \{ \log \left[ 1 + 6(1 - r)t^{-1} \right] \}^p \, dt \\
\leq 6^p (1 - r)^p \int_{1 - r}^1 t^{-p} \, dt \leq C(p)(1 - r), \\
J_3 \leq \int_{1}^\infty t^{-2} \{ \log \left[ 1 + 6(1 - r) \right] \}^p \, dt \leq 6^p (1 - r)^p,
\]

and so from (6)

\[
\int_{S(r)} [G(X, Y)]^p \, d\hat{\sigma}(X) \leq C(p)(1 - r).
\]

Hence

\[
(1 - |Y|)^{-\alpha} \mathcal{M}_p(G(\cdot, Y); r) \leq \frac{3}{4} (1 - r)^{-1/p} \mathcal{M}_p(G(\cdot, Y); r) \leq C(p),
\]

as required.

3. Proof of Theorem 1. Let \( 1 < p < (n - 1)/(n - 2) \), so that \( 0 < \alpha < 1 \), and let \( \mu \) be the measure corresponding to the potential \( v \). By Minkowski's inequality,

\[
\mathcal{M}_p(v; r) = \left\{ \int_{S(r)} \left[ \int_{B(1)} G(X, Y) \, d\mu(Y) \right]^p \, d\hat{\sigma}(X) \right\}^{1/p} \\
\leq \int_{B(1)} \mathcal{M}_p(G(\cdot, Y); r) \, d\mu(Y).
\]

Thus, from Lemmas 1 and 2,

\[
(1 - r)^\alpha \mathcal{M}_p(v; r) \leq o(1) + C(n, p) \int_{E(r)} (1 - |Y|) \, d\mu(Y) = o(1) \quad (r \to 1-)
\]

in the light of (3).
4. Proof of Theorem 2.

4.1. If \( \mu \) is a measure on \( B(1) \) satisfying (3), we define a finite measure \( \mu^* \) on \( B(1) \) by \( d\mu^*(Y) = (1 - |Y|) d\mu(Y) \). For any interval \( I \subseteq [0, 1) \), let \( A(I) = \{X : |X| \in I\} \). The maximal function \( M(d\mu^*) \) is defined by

\[
M(d\mu^*)(t) = \sup_{I \ni t} \mu^*(A(I))/|I|,\]

where \( |I| \) denotes the Lebesgue measure of \( I \). Also, for each \( k \in \mathbb{N} \), we define the interval \( I_k = [1 - 2^{-k}, 1 - 2^{-k-1}) \), and denote by \( \mu_k^* \) the restriction of \( \mu^* \) to \( A(I_k) \).

Well-known estimates for the maximal function yield the following lemma (cf. \([5, p. 453]\)).

**Lemma A.** For each \( k \in \mathbb{N} \), there exists \( r_k \in (1 - 2^{-k}/5, 1 - 2^{-k}/3) \) such that

\[
M(d\mu_k^*)(r_k) < 2^{k+4} \mu_k^*(A(I_k)).
\]

4.2. We now prove Theorem 2. Let \( n > 3 \) and \((n-1)/(n-2) < p < (n-1)/(n-3)\), so that \( 1 < a < 2 \), and let \( \mu \) be the measure corresponding to the potential \( v \). We give below the argument for \( 1 < a < 2 \), the case \( a = 1 \) being similar.

As in §3,

\[
(1 - r)^a \mathcal{M}_p(v; r) \leq o(1) + C(n, p) \int_{E(r)} \left[ \frac{1 - r}{|Y| - r} \right]^{a-1} d\mu^*(Y)
\]
as \( r \to 1^- \). Now let \((r_k)\) be a sequence as in Lemma A. Fixing \( k \) and approximating \((1 - r_k)/|x - r_k|\) by a monotone increasing sequence of step functions of the form

\[
f(x) = \sum_{j=1}^N a_j \chi_{I_j}(x),
\]

where each \( a_j \) is nonnegative and each interval \( I_j \) is symmetric about \( r_k \) it follows that

\[
\int_{E(r_k)} \left[ \frac{1 - r_k}{|Y| - r_k} \right]^{a-1} d\mu^*(Y) \leq 2M(d\mu_k^*)(r_k)(1 - r_k)^{a-1} \int_0^{(1-r_k)/4} x^{1-a} dx
\]

\[
= C(n, p)(1 - r_k)M(d\mu_k^*)(r_k).
\]

Since, from Lemma A,

\[
(1 - r_k)M(d\mu_k^*)(r_k) \leq 2^4 \mu_k^*(A(I_k)),
\]

it now follows from (7) that

\[
(1 - r_k)^a \mathcal{M}_p(v; r_k) \leq o(1) + C(n, p)\mu^*(A(I_k)) = o(1) \quad (k \to \infty),
\]
as required.

5. Examples 3 and 4.

5.1. Let \( h \) be as in Example 3 and \( p \geq 1 \). Then

\[
\int_{S(r)} [h(X)]^p d\sigma(X) = C(n)(1 - r^2)^p \int_0^\pi \sin^{n-2} \theta \{(1 - r \cos \theta)^2 + (r \sin \theta)^2\}^{-np/2} d\theta
\]

\[
= C(n)(1 - r^2)^p \int_0^\infty t^{n-2}(1 + t^2)^{-n+np/2}\{(1-r)^2 + (1+r)^2t^2\}^{-np/2} dt.
\]
Splitting up the integral in (8) into the components $J_1, J_2$ corresponding to the intervals $[0, 1], [1, \infty)$, it is clear from dominated convergence that $J_2(r)$ has a finite limit as $r \to 1^-$. Also

$$(1 - r)^{n\beta + 1 - n} J_1(r) = (1 + r)^{1 - n} \int_0^{\tan^{-1}((1+r)/(1-r))} \sin^{n-2} \phi \cos^{n(p-1)} \phi \times \{1 + (1 - r)^2 (1 + r)^2 \tan^2 \phi \}^{1-n+np/2} d\phi$$

has a finite positive limit as $r \to 1^-$, so

$$(9) \quad (1 - r)^{(n-1)(1-1/p)} \mu_p(h; r) \to C(n, p) \quad (r \to 1^-).$$

5.2. By Minkowski's inequality it is sufficient to show that

$$(10) \quad (1 - r)^{(n-1)(1-1/p)} \mu_p(h; r) \to 0 \quad (r \to 1^-)$$

for any positive harmonic function $h$ in $B(1)$ which does not majorize any positive multiple of a Poisson kernel. For simplicity we show this when $n = 2$; the higher dimensional argument requires minor modification.

Let $\mu$ be the measure on $[0, 2\pi)$ associated with $h$ in the Poisson integral representation. From §5.1 it is straightforward to deduce that

$$\limsup_{r \to 1^-} \frac{1 - r)^{p-1}}{\mu_p(h_r; r)} \leq C(p) \mu([\alpha, \beta])^p$$

for $0 < \alpha < \beta < 2\pi$. Let $k$ be a positive integer. No singleton has positive measure, so we can let $0 = \gamma_0 < \gamma_1 < \cdots < \gamma_k = 2\pi$ be such that

$$\mu([\gamma_{i-1}, \gamma_i]) = \mu([0, 2\pi))/k$$

for each $i \in \{1, \ldots, k\}$. Then

$$\limsup_{r \to 1^-} \frac{1 - r)^{p-1}}{\mu_p(h_r; r)} \leq C(p) \mu([0, 2\pi]))^p k^{1-p}.$$ Since $k$ can be arbitrarily large, (10) holds for any $p > 1$.

5.3. To establish Example 4, let $\varepsilon > 0$ and $p \geq 1$, and choose $\beta \in (1 - \varepsilon/n, 1)$. If $h$ is as in Example 3, it is easy to see that $h^\beta$ is a potential in $B(1)$. As in (8), we have

$$\mu_p(h^\beta; r)^p \geq C(n, p)(1 - r)^{\beta p} \int_0^{1-r} t^{n-2} \{1 - (1 - r)^2 + t^2\}^{-np\beta/2} dt$$

$$= C(n, p)(1 - r)^{(n-1)(1-p\beta)} \int_0^{\pi/4} \sin^{n-2} \phi \cos^{n(p\beta-1)} \phi d\phi,$$

and so

$$(1 - r)^{(n-1)(1-1/p)-\varepsilon} \mu_p(h^\beta; r) \geq C(n, p)(1 - r)^{-\varepsilon/n} \to +\infty \quad (r \to 1^-)$$

as claimed.
REFERENCES


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