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ON SULLIVAN'S INVARIANT MEASURE PROBLEM

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ABSTRACT. Sullivan has posed an invariant measure problem for which a positive answer is very plausible. It also seems highly plausible that hyperfinite AW^*-factors are injective. Surprisingly, it turns out that one of these problems must have a negative solution. Specifically we show that either the hyperfinite Takenouchi-Dyer factor is not injective, or no G-invariant category measure exists for any free, ergodic action of a countable nonamenable group G on a perfect Polish space.

Introduction. Let T be a complete metric space with no isolated points (for example $\mathbf{R}^n$). Let F_2 be the free group on two generators and let β be a free action of F_2 as homeomorphisms of T. Suppose that for some $t_0 \in T$ the orbit $\{\beta_g(t_0): g \in F_2\}$ is dense in T. Some time ago Professor Dennis Sullivan posed the following problem,

(1) *Can we choose T and β so that there exists a finitely additive probability measure, μ, on the Borel sets of T where μ vanishes on the meagre subsets of T and μ is F_2-invariant?*

Since F_2 can be densely embedded in the compact group $SO(3)$ [**2**], if we take T to be $SO(3)$ and let λ be Haar measure on the Borel sets of this compact group, then λ is F_2-invariant. But, of course, λ does not vanish on all meagre subsets of T. However this example suggests that it would be reasonable to hope for a positive answer.

The C^*-algebra of bounded Borel functions on the unit interval, modulo the ideal of functions with meagre support, is known as the Dixmier algebra, $\mathscr{D}$. Given any countable group G and any free ergodic action, β, of G on $\mathscr{D}$, there can be constructed a "monotone cross-product" algebra $M(\mathscr{D}, \beta, G)$ which is a monotone complete C^*-algebra with trivial centre, that is, a factor. Since it contains $\mathscr{D}$ as a maximal abelian *-subalgebra, $M(\mathscr{D}, \beta, G)$ is not a von Neumann algebra.

In surprising contrast to the situation for von Neumann algebras it was proved in [**12**] that all factors of the form $M(C(S), \beta, G)$ are, in fact, isomorphic. Let W be this canonical factor. In particular, the factors constructed by Dyer [**7**, **1**] and by Takenouchi [**7**, **13**] are both isomorphic to W. It is natural to ask:

(2) *Is W an injective C^*-algebra?*

Since W is easily seen to be hyperfinite, analogy with von Neumann algebras makes it very plausible to conjecture that the answer is "yes".

We show that Sullivan's Problem, Question 1, has a positive answer if, and only if, there exists a conditional expectation from W onto a subalgebra $M(F_2)$, where

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$M(F_2)$ can be identified with the von Neumann algebra generated by the left regular representation of F_2.

Although it is plausible to conjecture that both Question 1 and Question 2 have positive answers, we show in Theorem 2.3 that this is impossible. Either Question 1 or Question 2 must have a negative answer. We show that:

Either the hyperfinite algebra W is not injective, or no G-invariant category measure exists for any free, ergodic action of a countable nonamenable group G on a perfect Polish space.

1. Preliminaries: category measures and group actions. Let T be a perfect Polish space, that is, a topological space which is homeomorphic to a complete, separable metric space with no isolated points. Let $\mathrm{Bor}(T)$ be the σ-field of Borel subsets of T and let $M_g(T)$ be the σ-ideal of all meagre Borel subsets of T. A positive, finitely additive measure μ on $\mathrm{Bor}(T)$ is said to be a *category measure* if $\mu M = 0$ for all $M \in M_g(T)$ and $\mu T = 1$.

Let S be the Stone structure space of the Boolean algebra $\mathrm{Bor}(T)/M_g(T)$. Then S is also the Stone structure space of $\mathrm{Bor}(\mathbf{R})/M_g(\mathbf{R})$ because, for each perfect Polish space T, the Boolean algebras $\mathrm{Bor}(T)/M_g(T)$ and $\mathrm{Bor}(\mathbf{R})/M_g(\mathbf{R})$ are isomorphic, see [**11**, p. 155]. The commutative C^*-algebra $C(S)$ is called the *Dixmier algebra.* A discussion of properties of S is given in §0 of [**12**].

It is easy to see that category measures on T are in bijective correspondence with states on $C(S)$.

Let θ be a bijection of T onto T. Then, as in [**12**], θ is said to be a *pseudo-homeomorphism* if θ is a Borel bijection such that both θ and θ^{-1} map $M_g(T)$ onto $M_g(T)$. Clearly all homeomorphisms of T onto T are pseudo-homeomorphisms.

When θ is a pseudo-homeomorphism of T, there exists a dense G_δ-subset T_0 such that the restriction of θ to T_0 is a homeomorphism of T_0 onto T_0. This follows by applying a theorem of Kuratowski [**5**, p. 400] to θ and θ^{-1}.

When θ is a pseudo-homeomorphism of T, then it induces an automorphism of $\mathrm{Bor}(T)/M_g(T)$ which in turn induces a homeomorphism of S which then induces a *-automorphism of $C(S)$. Conversely, by a theorem of Maharam and Stone [**7**], every *-automorphism of $C(S)$ arises in this way from a pseudo-homeomorphism of T.

Throughout this paper, G is a countable group. Let X be either a Polish space or S or a dense G_δ-subset of S. An *action* of G on X is an homomorphism α from G into the group of all homeomorphisms of X onto itself. An action α is said to be *free* if, for each $g \in G$, other than the identity, the set of fixed points of α_g is a meagre subset of X. Following [**12**], we call α a (*generically*) *ergodic* action on X if, for some $x_0 \in X$, the orbit $\{\alpha_g(x_0) \colon g \in G\}$ is dense in X. By Lemma 1.1 of [**12**] α is a generically ergodic action on X if, and only if, the only Borel subsets of X which are G-invariant are either meagre or comeagre.

Sullivan's Problem, Question 1, can be put in the following slightly more general form.

Let G be a nonamenable countable group. Does there exist a perfect Polish space T, a free ergodic action α of G on T and a category measure μ on T such that μ is G-invariant with respect to the action α?

It follows from our earlier remarks that the above question can be reformulated as follows.

Let G be a nonamenable countable group. Let α be a free ergodic action of G on $C(S)$. Does there exist a state ϕ of $C(S)$ such that $\phi = \phi \circ \alpha_g$ for all $g \in G$?

Here it should be noted that α is a *free* action on $C(S)$ if, for each g in G, other than the identity, there does not exist a projection e in $C(S)$ such that the restriction of α_g to $eC(S)$ is the identity. The action is *ergodic* if the only G-invariant projections in $C(S)$ are 0 and 1.

2. Monotone cross-products. Saitô [**8**] gives a very clear account of monotone cross-products. He only discusses cross-products by abelian groups but everything extends to nonabelian (countable) groups without difficulty. Detailed information on the Hamana tensor product may be found in [**3**, **4**, **9**], see also the discussion in §3 of [**12**].

Let us recall that the elements of the Hamana tensor product

$$C(S)\overline{\otimes}\mathscr{L}(l^2(G))$$

correspond to those infinite matrices over $C(S)$ of the form $[a_{\gamma,\sigma}]$ $(\gamma \in G, \sigma \in G)$, such that: (i) For each $s \in S$, $[a_{\gamma,\sigma}(s)]$ corresponds to a bounded operator on $l^2(G)$, with respect to the natural orthonormal basis for $l^2(G)$.

(ii) The function $s \to [a_{\gamma,\sigma}(s)]$ is a σ-weakly continuous map from S into $\mathscr{L}(l^2(G))$.

(Warning: The definition of multiplication is not straightforward).

Let β be a free, ergodic action of G on $C(S)$. Then the monotone cross-product algebra $M(C(S), \beta, G)$ may be defined as that subalgebra of $C(S)\overline{\otimes}\mathscr{L}(l^2(G))$ for which the representing matrices $[a_{\gamma,\sigma}]$ $(\gamma \in G,\ \sigma \in G)$ satisfy the constraints $\beta_\rho(a_{\sigma\gamma}) = a_{\sigma\rho,\gamma\rho}$ for all σ, γ, ρ in G.

Let $M(G)$ be the subalgebra of $M(C(S), \beta, G)$ corresponding to those matrices $[t_{\sigma,\gamma}]$ in $M(C(S), \beta, G)$ where each $t_{\sigma,\gamma}$ is a scalar. Then

$$t_{\sigma,\gamma} = \beta_\rho(t_{\sigma,\gamma}) = t_{\sigma\rho,\gamma\rho}$$

for each σ, γ, ρ in G. Hence $M(G)$ can be identified with the von Neumann algebra generated by the left regular representation of G, see Sakai [**10**, p. 182].

As remarked in the introduction, Theorem 3.4 of [**12**] gives

LEMMA 2.1. *Let G_1 and G_2 be countable groups. Let β_1 and β_2 be free ergodic actions on $C(S)$ of, respectively, G_1 and G_2. Then*

$$M(C(S), \beta_1, G_1) \approx M(C(S), \beta_2, G_2).$$

We shall denote the canonical algebra $M(C(S), \beta, G)$ by "W".

Let us recall that a linear map Γ from a unital C^*-algebra A into A is said to be a *conditional expectation* when the following conditions are satisfied.

(i) The range of Γ is a unital C^*-subalgebra of A, say, B, with the same unit as A.

(ii) Γ is idempotent.

(iii) Γ is completely positive.

When Γ is a conditional expectation, $\Gamma 1 = 1$ and so, by (iii), $\|\Gamma\| = 1$, see Corollary 3.8 [**14**, p. 199]. It then follows from (i), (ii) and [**15**] that Γ is a bimodule homomorphism onto B. That is, for any b_1, b_2 in B and any $a \in A$,

$$\Gamma(b_1 a b_2) = b_1(\Gamma a)b_2.$$

THEOREM 2.2. *Let G be a countable group and let β be a free, ergodic action of G on $C(S)$. Then $W = M(C(S), \beta, G)$. There exists a G-invariant state on $C(S)$, for the action β, if, and only if, there exists a conditional expectation Γ from W onto $M(G)$, which may be identified with the von Neumann algebra generated by the left regular representation of G.*

PROOF. First suppose there exists a G-invariant state ϕ on $C(S)$. Then, by Lemma 3.5 of [**3**], $\phi\overline{\otimes}\,\mathrm{id}$ is a completely positive map from $C(S)\overline{\otimes}\mathscr{L}(l^2(G))$ onto $C\overline{\otimes}\mathscr{L}(l^2(G))$. Moreover

$$\phi\overline{\otimes}\,\mathrm{id}([a_{\gamma\sigma}]) = [\phi(a_{\gamma\sigma})].$$

Let Γ be the restriction of $\phi\overline{\otimes}\,\mathrm{id}$ to $M(C(S), \beta, G)$. When $[a_{\gamma,\sigma}]$ is in $M(C(S), \beta, G)$ we have $\beta_\rho(a_{\gamma,\sigma}) = a_{\gamma\rho,\sigma\rho}$ for all γ, σ, ρ in G.

Then, because of the G-invariance of ϕ,

$$\phi(a_{\gamma\rho,\sigma\rho}) = \phi(a_{\gamma,\sigma}).$$

Hence $[\phi(a_{\gamma,\sigma})]$ corresponds to an element of $M(G)$. So Γ maps $M(C(S), \beta, G)$ onto $M(G)$ and Γ is clearly idempotent. Thus Γ is a conditional expectation from W onto $M(G)$.

Conversely, let us assume that there exists a conditional expectation Γ from W onto $M(G)$.

Since G is a countable group and $M(G)$ may be identified with the von Neumann algebra generated by the left regular representation of G, there exists a faithful tracial state τ on $M(G)$, see Sakai [**10**, p. 182].

By the cross-product construction, there exists a maximal abelian *-subalgebra $\mathscr{D}$ of W, where $\mathscr{D}$ is *-isomorphic to $C(S)$ and on identifying $\mathscr{D}$ with $C(S)$, the automorphism β_γ of $\mathscr{D}$ is implemented by a unitary u_γ on $M(G)$. So, for each $a \in \mathscr{D}$ and $\gamma \in G$, $\beta_\gamma(a) = u_\gamma a u_\gamma^*$.

Let $\phi(a) = \tau\Gamma(a)$ for all $a \in \mathscr{D}$. Then

$$\phi\beta_\gamma(a) = \tau\Gamma(u_\gamma a u_\gamma^*).$$

Since Γ is a conditional expectation onto $M(G)$ and $u_\gamma \in M(G)$,

$$\Gamma(u_\gamma a u_\gamma^*) = u_\gamma(\Gamma a)u_\gamma^*.$$

Since τ is a tracial state on $M(G)$ we have

$$\tau\Gamma(u_\gamma a u_\gamma^*) = \tau(u_\gamma(\Gamma a)u_\gamma^*) = \tau\Gamma a.$$

Thus

$$\phi\beta_\gamma(a) = \phi(a) \quad \text{for all } a \in \mathscr{D} \text{ and all } \gamma \in G.$$

So there exists a G-invariant state on $C(S)$.

We come now to the main theorem.

THEOREM 2.3. *Either the hyperfinite AW^*-factor W is not injective, or whenever G is a nonamenable countable group, there does not exist a G-invariant state on $C(S)$ for any free ergodic action of G on $C(S)$.*

PROOF. Let us assume that the statement of the theorem is false. So we assume that W is injective and, for some free ergodic action of G (where G is nonamenable), there exists a G-invariant state on $C(S)$. So, by Theorem 2.2, there exists a conditional expectation Γ from W onto $M(G)$.

We embed W in $\mathscr{L}(H)$, where H is a Hilbert space of sufficiently large dimension. Since W is injective, there exists a conditional expectation Λ from $\mathscr{L}(H)$ onto W. Then $\Gamma\Lambda$ is a conditional expectation from $\mathscr{L}(H)$ onto $M(G)$. So $M(G)$, the von Neumann algebra of the left regular representation of G, is injective. Hence, by a slight modification of the proof of Theorem 4.2 [**6**], G is amenable. This contradiction establishes the theorem.

The powerful methods of Hamana have produced substantial advances and enabled him to obtain a number of deep results on AW^*-algebras. In [**4**] he formulated a theorem which would imply that W is injective. Unfortunately, there appears to be a subtle flaw in the proof of Theorem 3.3 [**4**]. The difficulty occurs in Lemma 3.2 [**4**], where the proof is valid when card I is countable but, for the application to Theorem 3.3 [**4**], it seems necessary to allow I to be uncountable. If this difficulty could be resolved and the injectivity of W established then this, together with the main theorem given here, would completely settle Sullivan's problem. It would show that invariant category measures do not exist for any free ergodic action of any nonamenable countable group on any perfect Polish space.

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