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ESTIMATES FOR GREEN'S FUNCTIONS

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ABSTRACT. Let $l_q = -\nabla^2 + q(x)$, $x \in R^3$, $0 \le q \le c(1+|x|)^{-a}$, $a > 2$, $l_q G_q(x,y) = \delta(x-y)$. If $q \ge p \ge 0$, $q \not\equiv p$, then $c|x-y|^{-1} < G_q(x,y) < G_p(x,y) \le (4\pi|x-y|)^{-1}, x \ne y$, for some positive $c = c(q)$. If $p \not\equiv 0$ then $G_p < (4\pi|x-y|)^{-1}$, $x \ne y$.

Introduction. If A and B are linear selfadjoint operators on a Hilbert space H,

$$(*) \qquad A \ge B > 0,\ D(A) \subset D(B), \quad \text{then } 0 < A^{-1} \le B^{-1}.$$

Here $A \ge B$ means that $(Af,f) \ge (Bf,f)$ for all $f \in D(A)$. However, if A^{-1} and B^{-1} are integral operators, the inequality $(*)$ does not imply that

$$(**) \qquad G_B(x,y) \ge G_A(x,y) \quad \text{for all } x \text{ and } y,$$

where $G_A(x,y)$ is the kernel of A^{-1}. This is well known and can be easily seen already in the example when A^{-1} and B^{-1} are matrices (e.g. if

$$A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B^{-1} = \begin{pmatrix} 5 & -1 \\ -1 & 3 \end{pmatrix},$$

then $B^{-1} \ge A^{-1}$ but the inequality $B^{-1}_{jm} \ge A^{-1}_{jm}$ does not hold, where B^{-1}_{jm} are the entries of the matrix B^{-1}).

It is therefore of interest to give conditions under which $(**)$ holds. For matrices such conditions are known, the corresponding matrices are called M-matrices (see [**1**, **2**]). For positive definite elliptic differential operators of second order in $L^2(D)$, where D is a bounded domain, it is established in [**3**] that their Green's functions are nonnegative. See also a discussion in [**7**, p. 209] of Beurling-Deny criteria for a selfadjoint operator to generate a positivity preserving semigroup. For higher order elliptic positive definite operators (such as biharmonic operator, for example) Green's function is not necessarily pointwise positive (some references can be found in [**3**]).

Formulation of the results. Assume that A is selfadjoint in $L^2 = L^2(R^3)$ operator defined by the expression $Au = -\nabla^2 u + q(x)u$ on $D(A) = H^2(R^3)$, where $H^2(R^3)$ is the standard Sobolev space, and

$$(1) \qquad 0 \le q(x) \le c(1+|x|)^{-a}, \qquad a > 2.$$

By c we denote various positive constants, $|x|$ is the length of the vector $x \in R^3$. Let

$$(2) \qquad 0 \le p(x) \le q(x)$$

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and let $G_q(x,y)$ denote Green's function:

$$[-\nabla^2 + q(x)]G_q(x,y) = \delta(x-y). \tag{3}$$

LEMMA 1. *If* (1) *and* (2) *hold then*

$$(4\pi|x-y|)^{-1} \geq G_p(x,y) \geq G_q(x,y) > 0 \quad \textit{for all } x, y \in R^3. \tag{4}$$

If $p \not\equiv 0$ *and* $q \not\equiv p$, *then*

$$(4\pi|x-y|)^{-1} > G_p(x,y) > G_q(x,y) > 0, \quad \textit{for } x \neq y. \tag{4'}$$

REMARK 1. Estimates of the type (4) are useful in applications [**5**, pp. 309–317, **6**].

PROOF. First let us prove the right inequality (4) using the strong maximum principle [**4**]. Suppose that $G_q(x_0,y_0) := G(x_0,y_0) \leq 0$ for some x_0 and y_0. Clearly $x_0 \neq y_0$ since $G(x,y) \to +\infty$ as $x \to y$. Since $G(x,y) \to 0$ as $|x-y| \to \infty$, one concludes that the function $G(x,y_0) := u(x)$ attains a nonpositive minimum at a point $\xi \neq y_0$. Consider the function $u(x) := G(x,y_0)$ in a ball $B_a = \{x\colon |\xi - x| \leq a\}$ which does not contain y_0. This function solves the equation

$$\nabla^2 u(x) - q(x)u(x) = 0 \quad \text{in } B_a. \tag{5}$$

Since $q(x) \geq 0$, we conclude that u cannot attain in B_a a nonpositive minimum at the point ξ unless $u \equiv \text{const}$ for $x \neq y_0$. The function $u(x) = G(x,y_0) \not\equiv \text{const}$: if $u \equiv \text{const}$ then the property $G(x,y_0) \to \infty$ as $x \to y_0$ cannot hold. This contradiction proves that $G_q(x,y) > 0$ for all x, y.

The left inequality (4) follows from the equation

$$G_p(x,y) = g(x,y) - \int_{R^3} g(x,z)p(z)G_p(z,y)\,dz, \qquad g := (4\pi|x-y|)^{-1}. \tag{6}$$

Since $p \geq 0$ and $g \geq 0$ one concludes that $G_p \leq g$.

In order to prove that $G_p \geq G_q$, define

$$v := G_p(x,y) - G_q(x,y). \tag{7}$$

Subtract from equation (3) the equation for G_p to get

$$\nabla^2 v - pv = (p-q)G_q \leq 0 \tag{8}$$

where assumption (2) was used. Since $v \to 0$ as $|x-y| \to \infty$ and v is bounded on compact sets, the strong maximum principle says that v cannot attain a nonpositive minimum, therefore, $v > 0$. Thus $G_p > G_q$.

Note that in order that Green's function $G_q(x,y)$ in Lemma 1 be positive, it is not necessary that $q(x)$ be nonnegative. Indeed, the following lemma holds. Let $a(x) \geq 0$, $a_M(r) = \sup_{|x|=r} a(x)$, $a(\infty) = 0$, $a \in L^\gamma(R^3)$, $\gamma > \frac{3}{2}$.

LEMMA 2. *If for some* $u_0(x) \geq c(|x|+1)^{-\alpha}$, $0 < \alpha < 1$, $c = \text{const} > 0$, *the equation* $(-\Delta - a(x))u_0 = 0$ *holds in* R^3, *then* $G_q(x,y) > 0$ *in* R^3 *provided that* $q(x) \geq -a(x)$. *If* $0 \geq q \geq -a(x)$ *and* $q(x) \not\equiv 0$ *then* $G_q(x,y) > (4\pi|x-y|)^{-1}$, $x \neq y$.

PROOF. Define V by the equation $G_q(x,y) = u_0(x)V(x,y)$. Then

$$\Delta_x G_q(x,y) = V\Delta_x u_0 + 2\nabla_x u_0 \cdot \nabla_x V + u_0 \Delta_x V(x,y).$$

Substitute this into the equation $-\Delta G_q + qG_q = \delta(x-y)$ to get for fixed y

$$-u_0\Delta V - 2\nabla u_0 \cdot \nabla V + (-\Delta u_0 + qu_0)V = \delta(x-y). \tag{9}$$

Note that V is by definition positive in a neighborhood of y. Therefore it is sufficient to prove that $V > 0$ outside of this neighborhood. This is done by proving that V cannot attain a nonpositive value outside of the above neighborhood. If it does, then the maximum principle says that $V \equiv \text{const}$ and this is a contradiction.

Since in (9) we have $-\Delta u_0 + qu_0 \geq -\Delta u_0 - a(x)u_0 = 0$, and $V = u_0^{-1}G_q = O(|x|^{\alpha-1})$ as $|x| \to \infty$, so that $|V| \to 0$ as $|x| \to \infty$, the maximum principle applies to V. Hence $V(x,y) > 0$ and this yields positivity of $G_q(x,y)$. To prove the last statement of Lemma 2, start with the formula $G_q = g - \int_{R^3} gqG_q\,dz$, which is analogous to (6), and note that the integral term is positive since $q \leq 0$, $q \not\equiv 0$, and $G_q > 0$, as we have already proved.

EXAMPLE. Let $|x| = r$, $u_0 := (r+\varepsilon)^{-\alpha}$, $0 < \alpha < 1$, $\varepsilon > 0$ is an arbitrary small number, $a(r) = (r+\varepsilon)^{-2}[\alpha(1-\alpha) + 2\varepsilon\alpha r^{-1}]$. If $q \geq -a(r)$ then, by Lemma 2, $G_q(x,y) > 0$. If $\alpha = \frac{1}{2}$ and $q \geq -(r+\varepsilon)^{-2}[\frac{1}{4} + \varepsilon r^{-1}]$ then $G_q > 0$. In particular, if $q \geq -(r+\varepsilon)^{-2}/4$ then $G_q > 0$. This result is close to the best possible as follows from Remark 2.

REMARK 2. Let $q_m(r) := \inf_{|x|=r} q(x)$, $q_M(r) := \sup_{|x|=r} q(x)$. The following result is known [**9**, §48]. If $x \in R^3$ and

$$\lim_{r\to\infty} \inf r^2 q_m(r) > -1/4$$

then the equation $-\Delta u + qu = 0$ does not have nontrivial solutions with nodal surface in a neighborhood of infinity, that is there is no nodal surface in the region $\Omega_R := \{x : |x| > R\}$ for some $R > 0$, (in this case the equation $-\Delta u + qu = 0$ is called nonoscillatory). If

$$\lim_{r\to\infty} \sup r^2 q_M < -1/4$$

then the equation $-\Delta u + q(x) = 0$ is oscillatory, that is any solution to this equation has nodal surfaces in any neighborhood Ω_R of infinity (i.e. for any $R > 0$ sufficiently large).

If the equation $(-\Delta - a(x))u = 0$ has a positive solution u_0, then this equation is not oscillatory. Since $(-a)_m = -a_M$ one has for $q = -a(x)$ the sufficient condition

$$\lim_{r\to\infty} \inf r^2 a_M(r) < 1/4,$$

for the equation $[-\Delta - a(x)]u = 0$ to be nonoscillatory. This condition generalizes Kneser's condition known in the one dimensional case as a sufficient condition for the equation $-u'' - a(x)u = 0$, $x \in R^1$, to be nonoscillatory. Practically one cannot take $a(x)$ large: if $a(x)$ were large, then the operator $-\nabla^2 - a(x)$ would have a negative eigenvalue $[-\nabla^2 - a(x)]\psi = -\lambda\psi$, $\lambda > 0$, where ψ decays exponentially at infinity. The corresponding eigenfunction is strictly positive, and the equation $[-\Delta - a(x)]u = 0$ cannot have positive solution which grows not faster than a power of $|x|$ at infinity. Indeed, if $[-\Delta - a(x)]\psi = -\lambda\psi$, $\psi > 0$, and $[-\Delta - a(x)]w = 0$, $w > 0$, then

$$-\lambda \int_{R^3} \psi w\,dx = \int_{R^3} w(-\Delta - a)\psi\,dx = \int_{R^3} [-\Delta - a(x)]w\psi\,dx = 0.$$

This is a contradiction since $\int \psi w\, dx > 0$. An integration by part was used and the integrals over the large sphere go to zero as the radius of the sphere grows since ψ decays exponentially at infinity.

Therefore Lemma 2 is useful practically in the case when $a(x)$ is a function for which the operator $-\nabla^2 - a(x)$ does not have negative eigenvalues and for which zero is a resonance with positive resonance function, that is with positive solution to the equation $(-\nabla^2 - a(x))w = 0$.

The following theorem gives a positive lower bound of Green's function in R^3.

THEOREM 1. *Under the assumption in Lemma* 1, *the estimates*

$$(4\pi|x-y|)^{-1} \geq G_p(x,y) \geq G_q(x,y) > c|x-y|^{-1}, \qquad x \neq y,$$

hold where $c = c_q > 0$, *and the inequalities are strict if* $p \not\equiv q$, $p \not\equiv 0$.

PROOF. Lemma 1 shows that the first two inequalities hold. The last inequality is proved in [**6**, p. 1341, formula (4)].

Extensions. In this section we extend the results in several directions.

(1) The result in Theorem 1 and its proof remain valid in R^d for $d \geq 3$ after the obvious modifications: $(4\pi|x-y|)^{-1}$ should be substituted by $\Gamma(d/2)|x-y|^{2-d}/2(d-2)\pi^{d/2}$ and $|x-y|^{-1}$ by $|x-y|^{2-d}$.

(2) General second order selfadjoint elliptic operators

$$Lu = -\sum_{j,m=1}^{d} \partial_j(a_{mj}(x)\partial_m u) + q(x)u,$$

where

$$\lambda \sum_{j=1}^{d} |t_j|^2 \leq \sum_{j,m=1}^{d} a_{mj}(x) t_m \bar{t}_j \leq \Lambda \sum_{j=1}^{d} |t_j|^2.$$

λ and Λ are positive constants which do not depend on x, $a_{mj}(x) \in C^1$ and $q(x)$ satisfies assumption (1), can be studied in place of $-\Delta + q(x)$.

(3) In this section we discuss the Green functions with boundary conditions on ∂D, where $D \subset R^d$, $d \geq 3$, D is a domain with smooth boundary ∂D, $\overline{D} = D \cup \partial D$. By $B_D u = 0$, $B_N u = 0$ and $B_R u = 0$ we mean $u|_{\partial D} = 0$, $\partial u/\partial n|_{\partial D} = 0$ and $(\partial u/\partial n + \sigma(x)u)|_{\partial D} = 0$ respectively, where $0 \leq \sigma(x)$ is smooth, n is the outer normal. By P_D, P_N, P_R we denote the set of functions $a(x) \in C(\overline{D})$ for which there exists a $u(x) \geq 0$ such that $(-\Delta + a(x))u(x) \geq 0$ in D and u satisfies $B_D u = 0$, $B_N u = 0$ or $B_R u = 0$ respectively. Let $\lambda_1(a)$ denote the first eigenvalue of $-\Delta + a(x)$ and $u_0(x)$ denote the nonnegative eigenfunction corresponding to $\lambda_1(a)$ under one of the three boundary conditions. We use P to denote one of the sets P_D, P_N or P_R in what follows. It is easy to see that $a(x) \in P$ if and only if $\lambda_1(a) \geq 0$.

REMARK 3. If $a(x) \geq 0$ then $u_0(x)|_{\partial D} > 0$ under Neumann or Robin boundary condition ($B_N u = 0$ or $B_R u = 0$).

PROOF. From the monotonicity of the dependence of first eigenvalue $\lambda_1(a)$ on $a(x)$, one concludes that $a(x) \in P$. Since $u_0(x) > 0$ in D, $u_0(x)|_{\partial D} \geq 0$. Suppose $u_0(s) = 0$ for some $s \in \partial D$. Since $-\Delta u_0 + a u_0 = \lambda_1(a) u_0 \geq 0$, the strong maximum principle and Hopf's lemma imply that $\partial u_0(s)/\partial n < 0$. This is a contradiction to the assumption $B_N u = 0$ or $B_R u = 0$.

LEMMA 3. *Let $a(x) \geq 0$ and continuous function $q(x) \geq a(x) - \lambda_1(a)$, then $q(x) \in P$. If $u(x) > 0$ for all $x \in D$ satisfies the inequality $(-\Delta + q)u(x) \geq 0$ and one of the boundary conditions $B_N u = 0$ or $B_R u = 0$, then $u(x)|_{\partial D} > 0$.*

PROOF. Since the first eigenvalue $\lambda_1(q)$ of $-\Delta + q(x)$ is not less than that of the operator $-\Delta + (a(x) - \lambda_1(a))$ which is $\lambda_1(a) - \lambda_1(a) = 0$, it follows that $q(x) \in P$.

Let $u_0(x)$ denote the (nonnegative) eigenfunction corresponding to $\lambda_1(a)$ with one of the conditions $B_N u = 0$ or $B_R u = 0$. Define the function $V(x)$ by the equation $u(x) = u_0(x)V(x)$. Then

$$0 \leq (-\Delta + q)u = -u_0 \Delta V - 2\nabla V \cdot \nabla u_0 + (q + \lambda_1(a) - a(x))u_0 V.$$

Note that $q + \lambda_1(a) - a(x) \geq 0$ by the assumption and that on boundary ∂D we have:

(i) In the case of Neumann condition $B_N u = 0$,

$$0 = \frac{\partial u}{\partial n} = V \frac{\partial u_0}{\partial n} + u_0 \frac{\partial V}{\partial n} = u_0 \frac{\partial V}{\partial n}.$$

One thus concludes by Remark 3 that $\partial V / \partial n = 0$.

(ii) In the case of Robin condition

$$0 = \frac{\partial u}{\partial n} + \sigma u = V \left(\frac{\partial u_0}{\partial n} + \sigma u_0 \right) + u_0 \frac{\partial V}{\partial n} = u_0 \frac{\partial V}{\partial n}.$$

One can also conclude that $\partial V / \partial n = 0$.

By the strong maximum principle, V either attains its nonpositive minimum on ∂D or it is a constant. Using Hopf's lemma, which says that at the point $s \in \partial D$ of minimum $\partial V(s)/\partial n < 0$, and the result $\partial V/\partial n = 0$ one concludes that V cannot attain its minimum on ∂D and therefore $V \equiv$ constant. Since $u_0(x) > 0$ in D, it follows that this $V \equiv \text{const} > 0$, and, by Remark 3, that $u(x) = u_0(x)V > 0$ for $x \in \partial D$.

We now discuss the estimates for Green functions under the Dirichlet, Neumann or Robin boundary conditions. By G_{qD}, G_{qR}, and G_{qN} we denote the corresponding Green functions. In the case of Neumann condition we always assume $q(x) \not\equiv 0$ so that the Green function G_{qN} is uniquely determined.

We then have the following propositions.

PROPOSITION 1. *If $q(x) \geq a(x) - \lambda_1(a)$ and $a(x) \geq 0$ then $G_q(x, y) > 0$ for $x, y \in D$.*

Let $a(x) \equiv 0$ and λ_1 be the first eigenvalue of the Laplacian $-\Delta$ under one of the three boundary conditions, Proposition 1 implies

COROLLARY. *$G_q(x, y) > 0$ for $x, y \in D$, provided that $q(x) \geq -\lambda_1$.*

PROOF OF PROPOSITION 1. Since $q(x) \in P$ by Lemma 3, it follows that $-\Delta + q(x)$ is nonnegative definite. Therefore the positivity of G_q follows from the Aronszajn-Smith theorem (see §5 in [**3**]).

One can also prove Proposition 1 by applying the strong maximum principle to the function $V(x, y)$, where $V(x, y)$ is defined by $G_q(x, y) = u_0(x)V(x, y)$, and using the argument in Lemma 3.

PROPOSITION 2. *If $q_1(x) \geq q_2(x) \geq a(x) - \lambda_1(a)$ for some $a(x) \geq 0$, and $q_1 \not\equiv q_2$, then $G_{q_2}(x,y) > G_{q_1}(x,y) > 0$ for $x \neq y$, $x, y \in D$.*

PROOF. Strict positivity of G_{q_i} is a result of Proposition 1. The first inequality follows from the equation

$$G_{q_2}(x,y) - G_{q_1}(x,y) = \int_D (q_1(z) - q_2(z)) G_{q_1}(x,z) G_{q_2}(z,y)\, dz$$

and from the positivity of G_{q_i}, $i = 1, 2$.

PROPOSITION 3. *If $q(x) \geq a(x) - \lambda_1(a)$ for some $0 \leq a(x) \in P$, then*

$$G_{qD}(x,y) < G_{qR}(x,y) < G_{qN}(x,y) \quad \text{for } x, y \in \overline{D},\ x \neq y, \tag{10}$$

and

$$0 < G_{qD}(x,y) \quad \text{for } x, y \in D. \tag{11}$$

PROOF. *Step* 1. Let us prove the first inequality in (10). The argument in the proof of Lemma 3 shows that $G_{qR}(x,y) > 0$ for x or y on ∂D. Thus $(-\Delta + q)(G_{qR} - G_{qD}) = 0$, $(G_{qR} - G_{qD})|_{\partial D} > 0$. Define V by the equation

$$G_{qR}(x,y) - G_{qD}(x,y) = u_0(x) V(x,y), \tag{12}$$

where $u_0(x)$ is the (positive) eigenfunction of $-\Delta + a(x)$ corresponding to the eigenvalue $\lambda_1(a)$ under Robin condition. One has

$$-u_0 \Delta_x V - 2\nabla_x V \cdot \nabla u_0 + (q(x) + \lambda_1(a) - a(x)) u_0 V = 0 \tag{13}$$

with $q + \lambda_1(a) - a(x) \geq 0$ and $V(x,y) > 0$ for x or y on ∂D. Note that $G_{qR}(x,y) - G_{qD}(x,y) = G_{qR}(x,y) > 0$ for x or y on ∂D and since $u_0 > 0$ on ∂D, formula (12) implies that $V(x,y) > 0$ for x or y on ∂D. The maximum principle implies that $V(x,y) > 0$ for $x, y \in \overline{D}$. Therefore

$$G_{qR}(x,y) > G_{qD}(x,y) \quad \text{for } x, y \in \overline{D}. \tag{14}$$

Step 2. Let us prove the second inequality in (10). Note that

$$\frac{\partial}{\partial n}(G_{qR} - G_{qN})\bigg|_{\partial D} = \frac{\partial}{\partial n} G_{qR}\bigg|_{\partial D} = -\sigma(x) G_{qR} \leq 0.$$

Define V by the equation

$$G_{qR}(x,y) - G_{qN}(x,y) = u_0(x) V(x,y) \tag{15}$$

where $u_0(x)$ is the (positive) eigenfunction of the operator $-\Delta + a(x)$ corresponding to Neumann boundary condition with first eigenvalue $\lambda_1(a)$. Then

$$\frac{\partial(u_0 V)}{\partial n}\bigg|_{\partial D} = \frac{\partial(G_{qR} - G_{qN})}{\partial N}\bigg|_{\partial D} \leq 0.$$

Thus $u_0(\partial_x V/\partial n)|_{\partial D} \leq 0$. Since $u_0(x)|_{\partial D} > 0$ by Remark 3, it follows that

$$\frac{\partial_x V}{\partial n}\bigg|_{\partial D} \leq 0. \tag{16}$$

Equation (13) and the strong maximum principle imply that $V(x,y)$ can attain its nonnegative maximum only at some x on ∂D for any given $y \in \overline{D}$. By

Hopf's lemma, $\partial_x V/\partial n|_{\partial D} > 0$. This contradicts to (16). Therefore $G_{qR}(x,y) - G_{qN}(x,y) < 0$ for $x, y \in \overline{D}$, $x \neq y$.

Note that there is no positive lower bound for Green's function $G_q(x,y)$ which is uniform in q. Indeed if $q(x) \to +\infty$ in $\widetilde{D} \subset D$, then $G_q(x,y) \to 0$ in $\widetilde{D}$ as follows from the results in [**8**].

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