ABSTRACT. This paper gives a counterexample to the conjecture that the continuity of the conjugate $\overline{f}$ of an $f \in C(T)$ implies the continuity of the best uniform approximation $g \in H^\infty(T)$ of $f$. It also states two conditions which imply the continuity of $g$.

Let $L^\infty(T)$ the space of bounded measurable functions on the unit circle $T$, $H^\infty(T)$ the subalgebra of $L^\infty(T)$ consisting of nontangential limits of bounded analytic functions in the unit disk and write $\|f\|_\infty$ for the (essential supremum) norm of $f \in L^\infty(T)$. Also, let $C(T)$ be the space of all continuous functions on $T$.

It is known that any $f \in L^\infty(T)$ has at least one best approximation $g \in H^\infty(T)$, in the sense that

$$d = \|f - g\|_\infty = \inf_{h \in H^\infty} \|f - h\|_\infty$$

and that, by duality

$$d = \sup \left\{ \left| \int_0^{2\pi} f(\theta)F(\theta) \frac{d\theta}{2\pi} \right| : F \in H^1(T), F(0) = 0, \|F\|_1 \leq 1 \right\}$$

where $H^p(T)$ $(0 < p < \infty)$ is the Hardy space of all nontangential limits of functions $F$ analytic in the unit disc such that

$$\|F\|_p^p = \sup_{0<r<1} \int_0^{2\pi} |F(re^{i\theta})|^p \frac{d\theta}{2\pi} < +\infty.$$
(b) if \( \tau \in [0, 2\pi] \) and if

\[
f_\tau(\theta) = f(\theta) - f(\tau), \quad g_\tau(\theta) = g(\theta) - f(\tau)
\]

then there is \( \delta > 0 \) and \( r_0 > 0 \) such that

\[
|g_\tau(z)| \geq \frac{1}{2} \cdot \|f - g\|_\infty \quad \text{on } W_\tau = \{z = re^{i\theta} : |\theta - \tau| < \delta, r_0 < r < 1\}
\]

where \( \delta \) and \( r_0 \) can be independent of \( \tau \).

We consider the problem of how the regularity of \( f \) affects the regularity of \( g \). In [1] the following is proved.

**Theorem 2.** If \( f \) is Dini-continuous, i.e. if \( \int_0^\infty (\omega(t)/t) \, dt < +\infty \), where \( \omega(t) = \sup_{|x - y| \leq t} |f(x) - f(y)| \) is the modulus of continuity of \( f \), then its best approximation \( g \) is also continuous.

In [1] a function \( f \) is constructed, continuous but not Dini-continuous, whose best approximation \( g \) is not continuous.

Because the Dini-continuity of \( f \) implies the continuity of its conjugate \( \bar{f} \) and because of the proof in [1], it was conjectured that, for \( f \in C(T) \), the continuity of \( \bar{f} \) and the continuity of \( g \) are equivalent.

It was proved by Sarason that the continuity of \( g \) does not imply the continuity of \( \bar{f} \). See [2, p. 177].

This paper provides a counterexample for the other half of the conjecture. It constructs a continuous function \( f \), whose conjugate \( \bar{f} \) is continuous, but whose best approximation \( g \) is not. We also give two further conditions on \( f \) which imply \( g \) is continuous.

In the following \( \bar{f} \) is the complex conjugate of \( f \).

**Theorem 3.** If \( \bar{f} \in A(T) = H^\infty(T) \cap C(T) \) and \( \int_0^\infty (\omega^2(t)/t) \, dt < +\infty \), then \( g \), the best approximation of \( f \), is continuous.

**Theorem 4.** If \( \bar{f} \in A(T) \) and \( |\bar{f}|^2 \in C(T) \) and \( \int_0^\infty (\omega^3(t)/t) \, dt < +\infty \), then \( g \) is continuous.

**Theorem 5.** There exists a function \( f \), such that \( \bar{f} \in A(T) \), but such that its best approximation \( g \) is not continuous.

Since \( \bar{f} = -if \) when \( \bar{f} \in A(T) \), the function in Theorem 5 has a continuous conjugate.

**Proof of Theorem 3.** Suppose \( \|f - g\|_\infty = 1 \). Fix \( \tau \in [0, 2\pi] \). Then, from Theorem 1(b), \( g_\tau(z) \) has a well-defined logarithm on \( W_\tau \), which is given by

\[
\log g_\tau(z) = \frac{1}{2\pi} \int_{|\theta - \tau| \leq \delta} \log |g_\tau(\theta)| \cdot \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\theta + R_\tau(z), \quad z \in W_\tau,
\]

where \( R_\tau(z) \) is the integral over \( |\theta - \tau| > \delta \) plus the logarithm of the inner factor of \( g_\tau \). Since \( |g_\tau| \geq \frac{1}{2} \) on \( W_\tau \), this inner factor is analytic across \( |\theta - \tau| < \delta \). So \( R_\tau(z) \) and its derivative are bounded on \( |z - e^{i\tau}| < \delta_1 \), for some \( \delta_1 < \delta \), independent of \( \tau \). This implies

\[
|R_\tau(z) - R_\tau(w)| \leq c|z - w| \quad \text{for } |z - e^{i\tau}| < \delta_1, \ |w - e^{i\tau}| < \delta_1.
\]
We also have

\[ |f(\theta) - g(\theta)| = 1 \quad \text{a.e.} \ (d\theta) \]

from which

\[ |f_r(\theta) - g_r(\theta)| = 1 \quad \text{a.e.} \ (d\theta) \]

and

\[ |g_r|^2 = 1 + 2 \cdot \Re(\overline{f_r} \cdot g_r) - |f_r|^2. \]

Therefore

\[
\log |g_r| = \frac{1}{2} \log |g_r|^2 = \frac{1}{2} [2 \cdot \Re(\overline{f_r} \cdot g_r) - |f_r|^2 + O(|f_r|^2)] \\
= \Re(\overline{f_r} g_r) + O(|f_r|^2),
\]

and

\[
\log g_r(z) = \frac{1}{2\pi} \int_{|\theta - \tau| \leq \delta} \Re(\overline{f_r} g_r) \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\theta \\
+ \frac{1}{2\pi} \int_{|\theta - \tau| > \delta} O(|f_r|^2) \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\theta + R^*_r(z).
\]

Since \( \overline{f_r} \) is analytic, \( \overline{f_r} g_r \) is also analytic, which implies that

\[
\frac{1}{2\pi} \int_0^{2\pi} \Re(\overline{f_r} g_r) \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\theta = \overline{f_r}(z) \cdot g_r(z).
\]

Thus:

\[
\log g_r(z) - \overline{f_r}(z) g_r(z) = \frac{1}{2\pi} \int_{|\theta - \tau| \leq \delta} O(|f_r|^2) \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\theta + R^*_r(z)
\]

where

\[ R^*_r(z) = R_r(z) - \frac{1}{2\pi} \int_{|\theta - \tau| > \delta} \Re(\overline{f_r} g_r) \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\theta \]

and so, by (2),

(3) \[ |R^*_r(z) - R^*_r(w)| \leq c|z - w| \quad \text{for} \ |z - e^{ir}| < \delta_1, \ |w - e^{ir}| < \delta_1. \]

If \( z \) is in a truncated cone \( \Gamma(\tau) \), which is inside \( |z - e^{ir}| < \delta_1 \) and has vertex \( e^{ir} \), then

\[
\left| \frac{e^{i\theta} + z}{e^{i\theta} - z} \right| < \frac{c}{|\theta - \tau|},
\]

and so

\[
|\log g_r(z) - \overline{f_r}(z) g_r(z) - R^*_r(z)| \leq c \cdot \int_0^\delta \frac{\omega^2(t)}{t} \, dt.
\]

Since \( \overline{f_r}(z) \to 0 \) as \( z \to e^{ir} \),

\[
|\log g_r(z) - R^*_r(z)| \leq c \int_0^\delta \frac{\omega^2(t)}{t} \, dt + \eta(\delta)
\]

where \( \eta(\delta) \to 0 \) as \( \delta \to 0 \). Hence, by (3),

\[ |g_r(z) - g_r(w)| \leq c|z - w| + \eta_1(\delta), \quad z, w \in \Gamma(\tau), \]

where \( \eta_1(\delta) \to 0 \) as \( \delta \to 0 \).
Now, if \( \sigma \) and \( \tau \) are close to each other and \( z \in \Gamma(\tau) \cap \Gamma(\sigma) \) then
\[
|g(e^{i\tau}) - g(e^{i\sigma})| \leq |g(e^{i\tau}) - g(z)| + |g(z) - g(e^{i\sigma})|
= |g_\tau(e^{i\tau}) - g_\tau(z)| + |g_\sigma(z) - g_\sigma(e^{i\sigma})|
\leq c|e^{i\tau} - z| + c|e^{i\sigma} - z| + 2n_1(\delta) \leq c|\tau - \sigma| + 2n_1(\delta),
\]
and
\[
\lim_{\sigma \to \tau} |g(e^{i\tau}) - g(e^{i\sigma})| \leq 2n_1(\delta)
\]
so that
\[
\lim_{\sigma \to \tau} g(e^{i\sigma}) = g(e^{i\tau})
\]
and \( g \) is continuous.

**Proof of Theorem 4.** Now we carry the expansion of \( \log |g_\tau| \) one step further:
\[
\log |g_\tau| = \frac{1}{2} \left[ 2 \text{Re}(\overline{g_\tau}g_\tau) - |f_\tau|^2 - \frac{(2 \text{Re}(\overline{f_\tau}g_\tau) - |f_\tau|^2)^2}{2} + O(|f_\tau|^3) \right]
= \text{Re}(\overline{f_\tau}g_\tau) - \frac{1}{2} |f_\tau|^2 - (\text{Re}(\overline{f_\tau}g_\tau))^2 + O(|f_\tau|^3)
= \text{Re}(\overline{f_\tau}g_\tau) - \frac{1}{2} |f_\tau|^2 - \frac{1}{2} \overline{f_\tau}g_\tau|^2 - \frac{1}{2} \text{Re}(\overline{f_\tau}g_\tau)^2 + O(|f_\tau|^3)
= \text{Re}(\overline{f_\tau}g_\tau) - \frac{1}{2} \text{Re}(\overline{f_\tau}g_\tau)^2 - \frac{1}{2} |f_\tau|^2
- \frac{1}{2} |f_\tau|^2 (1 + 2 \text{Re}(\overline{f_\tau}g_\tau) - |f_\tau|^2) + O(|f_\tau|^3)
= \text{Re}(\overline{f_\tau}g_\tau) - \frac{1}{2} \text{Re}(\overline{f_\tau}g_\tau)^2 - |f_\tau|^2 + O(|f_\tau|^3).
\]
Now, because
\[
\frac{1}{2\pi} \int_0^{2\pi} \text{Re}(\overline{f_\tau}g_\tau) e^{i\theta} + z \ d\theta = (\overline{f_\tau}(z)g_\tau(z))^2
\]
since \( \overline{f}_\tau \in H^\infty(T) \), we get
\[
\log g_\tau(z) - \overline{f_\tau}(z)g_\tau(z) + \frac{1}{2} (\overline{f_\tau}(z)g_\tau(z))^2
= -\frac{1}{2\pi} \int_{|\theta - \tau| \leq \delta} |f_\tau(\theta)|^2 e^{i\theta} + z \ d\theta + \frac{1}{2\pi} \int_{|\theta - \tau| \leq \delta} O(|f_\tau|^3) e^{i\theta} + z \ d\theta + R_\tau^{**}(z)
\]
where
\[
R_\tau^{**}(z) = R_\tau^{*}(z) + \frac{1}{2\pi} \int_{|\theta - \tau| > \delta} \text{Re}(\overline{f_\tau}g_\tau) e^{i\theta} + z \ d\theta
\]
and so
\[
|R_\tau^{**}(z) - R_\tau^{**}(w)| \leq c|z - w| \quad \text{for } |z - e^{i\tau}| \leq \delta_1, \ |w - e^{i\tau}| \leq \delta_1.
\]
Now, the continuity of \( |\overline{f}|^2 \) implies the continuity of \( |f_\tau|^2 \), and this implies the continuity of the first integral. The rest of the proof proceeds as in Theorem 3.

**Proof of Theorem 5.** Consider the function
\[
u(t) = -\alpha_1 \log |\log t|, \quad 0 < t < \frac{1}{2},
= -\alpha_2 \log |\log t|, \quad -\frac{1}{2} < t < 0,
\]
extended to be smooth in \([-\pi, \pi] - \{0\} \), and consider the harmonic extension \( u(z) \) of \( u(t) \) inside the unit disk, its conjugate \( \overline{u}(z) \) and \( f(z) = e^{u(z) - i\overline{u}(z)} \).
Then, since \( \tilde{u}(t) \) is continuous in \([-\pi, \pi] - \{0\} \), and \(|f(z)| = e^{u(z)} \to 0\) as \(z \to 1\), we see that \(\tilde{f} \in A(T)\).

If \(\frac{1}{3} < \alpha_1 \leq \frac{1}{2}\) and \(\frac{1}{2} < \alpha_2\), then
\[
\int_{0}^{1/2} \frac{|f(t)|^2}{t} \, dt < +\infty \quad \text{and} \quad \int_{-1/2}^{0} \frac{|f(t)|^3}{|t|} \, dt < +\infty.
\]
but
\[
\int_{0}^{1/2} \frac{|f(t)|^2}{t} \, dt = +\infty \quad \text{and} \quad \int_{-1/2}^{0} \frac{|f(t)|^2}{|t|} \, dt < +\infty.
\]
The last two imply that
\[
|f|^2(r) \to +\infty \quad \text{as} \quad r \to 1 - .
\]
From
\[
\log g(r) - \bar{f}(r)g(r) + \frac{1}{2}(\bar{f}(r) \cdot g(r))^2
= |f|^2(r) + i|f|^2(r) + \frac{1}{2\pi} \int_{|\theta - \tau| \leq \delta} O(|f|^3) \frac{e^{i\theta} + r}{e^{i\theta} - r} \, d\theta + R^{**}(r)
\]
we get that
\[
\arg g(r) \to +\infty \quad \text{as} \quad r \to 1 - .
\]
Thus \(g\) is not continuous.

REFERENCES


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