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A NEW PROOF OF AN INEQUALITY OF LITTLEWOOD AND PALEY

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ABSTRACT. A fairly elementary new proof is presented of the inequality ($p \geq 2$):

$$\int |h'|^p(1-|z|)^{p-1}\,dx\,dy \leq \|h\|^p_{H^p}, \qquad f \in H^p.$$

In addition, the inequality

$$\int |h|^{p-s}|h'|^s(1-|z|)^{s-1}\,dx\,dy \leq \|h\|^p_{H^p}$$

is shown to hold for $h \in H^p$, $p > 0$, if and only if $2 \leq s < p+2$, generalizing the known case $s = 2$.

1. Introduction. Let U denote the open unit disk in the complex plane with boundary $T = \{z \colon |z| = 1\}$. Let m denote normalized area measure on U (i.e. $r\,d\theta\,dr/\pi$) and let σ denote normalized arc length ($d\theta/2\pi$) on T.

The inequality of Littlewood and Paley referred to in the title is the one contained in the following thorem, proved by J. E. Littlewood and R. E. A. C. Paley in [**3**].

THEOREM A. *If f is a function in $L^p(T)$ for $p \geq 2$ and if u is the harmonic function on u defined via the Poisson integral of f, then*

$$\int |\nabla u(z)|^p(1-|z|^2)^{p-1}dm(z) \leq C^p \int |f|^p\,d\sigma \tag{1.1}$$

where C is a constant independent of f and p.

The usual method of proof is to apply the Riesz Convexity Theorem to the operator $f \mapsto \nabla u(z)(1-|z|^2)$ acting on functions f on the measure spaces (T,σ) and taking them to functions on $(U,(1-|z|^2)^{-1}dm(z))$. It is relatively easy to show that this operator is of type $(2,2)$ as well as (∞,∞) and the Riesz theorem produces (1.1). There is another proof outlined in [**3**] but it is, if anything, deeper than the one I have just described. Some time ago (in [**4**]) I made use of Theorem A to obtain estimates on integrals of the form $\int |h^{(n)}|^q\,d\mu$ where h is an analytic function in the Hardy space H^p, $q > p$ and μ is a positive measure on u. The method I used was to integrate local estimates of $|h^{(n)}(z)|$ in terms of an area average of $|h'|$ over a disk containing z. The result was $(\int |h^{(n)}|^q\,d\mu)^{1/q} \leq C(\int |h'|^p(1-|z|)^{p-1}dm)^{1/p}$ provided μ satisfied a certain simple inequality similar to Carleson's condition. It occurred to me at the time that I was simply swapping one measure μ for another, $(1-|z|)^{p-1}dm$, satisfying a similar condition and that a similar approach might

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be used (i.e. integration of local estimates) to get directly to $(\int |h^{(n)}|^q \, d\mu)^{1/q} \leq C(\int |h|^p \, d\sigma)^{1/p}$ without using Theorem A. This turned out to be the case but would have unnecessarily complicated the paper [**4**] and so it never appeared there. This approach produced (and is essentially equivalent to) a proof of Theorem A which is quite elementary and it is this proof that I will present here.

The method of proof I will present is based on two ideas. Firstly, the integrand $|\nabla u(z)|^p$ in (1.1) may be estimated in terms of an area average of some other function and secondly, a weighted area integral of this other function is equal to the L^p norm of the boundary function f. To apply this method we need an area integral that equals the L^p norm of f. This is provided by the following theorem, Theorem B. This theorem tends to get rediscovered from time to time. I had thought it part of the mathematical folklore, having found only a weakened version in [**6**]. I am grateful to the referee for pointing out the references discussed at the end of the proof.

THEOREM B. *If u is real valued and harmonic in a neighborhood of the closed disk $\overline{U}$, and h is holomorphic in such a neighborhood then*

(i) *if* $p > 1$

$$\int |u|^p \, d\sigma = |u(0)|^p + \frac{p^2 - p}{2} \int |\nabla u|^2 |u|^{p-2} \log \frac{1}{|z|} \, dm.$$

(ii) *if* $p > 0$

$$\int |h|^p \, d\sigma = |h(0)|^p + \frac{p^2}{2} \int |h'|^2 |h|^{p-2} \log \frac{1}{|z|} \, dm.$$

I will prove here only (ii) and merely indicate how (i) differs. If one assumes only that h is holomorphic in U (not $\overline{U}$) then the right side of (ii) becomes equal to $\lim_{r \to 1} \int |h(re^{i\theta})|^p \, d\sigma(\theta)$ which is, by definition, $\|h\|_{H^p}^p$ and its finiteness is the criterion for h to belong to the *Hardy space* H^p. For any function u on T we write $\|u\|_p^p = \int |u|^p \, d\sigma$. When h is analytic in a neighborhood of $\overline{U}$, we have $\|h\|_{H^p} = \|h\|_p$ and we will normally use only the latter expression.

PROOF OF THEOREM B. We make use of Green's formula:

$$\int_R (g\Delta v - v\Delta g) \, dx \, dy = \int_{\partial R} (g\partial v/\partial n - v\partial g/\partial n) \, ds$$

where ds denotes arc length integration. We apply this to the following circumstance $v = |h|^p$, $g = \log(1/|z|)$, $R = U \setminus \varepsilon\overline{U} - \bigcup_{k=1}^n \overline{D}_k$ where D_k is the disk of radius ε centered at a_k and $\{a_1, a_2, \ldots, a_n\}$ are the zeros of h in $\overline{U}$. We obtain

$$\pi \int_R p^2 |h|^{p-2} |h'|^2 \log \frac{1}{|z|} \, dm = 2\pi \int_T |h|^p \, d\sigma - \int_{\partial\varepsilon U} |h|^p \frac{ds}{\varepsilon} + o(1)$$

where $o(1)$ represents integrals around ∂D_k that go to 0 as $\varepsilon \to 0$. Since $R \to U$ as $\varepsilon \to 0$ and $\int_{\partial\varepsilon U} |h|^p \, ds/\varepsilon \to 2\pi |h(0)|^p$ we obtain (ii). The reason the integrals around ∂D_k tend to zero is that they ae dominated by

$$\sup_{D_k} \left(|h|^{p-1} |h'| \log \frac{1}{|z|} + |h|^p \frac{1}{\varepsilon} \right) 2\pi\varepsilon$$

which tends to zero because $|h|^{p-1}|h'| = O(\varepsilon^{np-1})$ if n is the order of zero of h at a_k. Similarly the integral over R tends to that over U because $|h|^{p-2}|h'|^2 \log(1/|z|)$ is integrable even near zeros of h (where it behaves like $|z-a_k|^{np-2}$).

In part (i) the same proof is used except the calculation of $\Delta|u|^p$ yields $(p^2-p)|u|^{p-2}|\nabla u|^2$ and the zero set of u is one-dimensional instead of 0-dimensional so the limit arguments require $p > 1$. $\square$

We will use Theorem B only in case $p \geq 2$. The problem of avoiding the zero sets of h and u need not even occur ($|u|^p$ and $|h|^p$ are twice continuously differentiable) and the proof is more elementary.

An equivalent version of Theorem B appears as equations 4.3 and 4.7 on p. 243 of P. Stein's paper [**8**]. To get the version presented here one has only to integrate those equations from 0 to 1. Stein's proof is in fact essentially equivalent to the one given here. Equation 4.3 of Stein's turns out to be a special case of an equation due to Hardy, so it is called the Hardy-Stein identity. Another version of Theorem B in a more general setting is Equation 10, p. 462 of C. S. Stanton's paper [**7**]. The right hand side of Theorem B (ii) is there expressed in the form $(p^2/2)\int w^{p-2}N_h(w)\,dm(w)$ where N_h is a "counting function". The change of variables $w = h(z)$ transforms this integral into B(ii). This was pointed out by J. Shapiro in [**6**].

The proof of Theorem A that I will present in the next section proceeds under the assumption that u is harmonic in a neighborhood of $\overline{U}$ so that the boundary function f is just $u|_T$. Those readers needing the greater generality stated in Theorem A may simply apply the weaker result to $u_r = u(rz)$ and use Fatou's theory to produce (1.1) in the limit as $r \to 1$.

The most advanced facts needed for my proof are (1) the representation of a harmonic function u in terms of its boundary values as $u(z) = u(0) + \sum_{n=1}^{\infty} u_n z^n + u_n \bar{z}^n$ where u_n, $n = \pm 1, \pm 2, \ldots$, are the Fourier coefficients of $u(e^{i\theta})$, (2) the orthogonality of exponentials $e^{in\theta}$ and Parseval's relationship $\|u\|_2^2 = |u(0)|^2 + \sum_{\infty}^{\infty} |u_n|^2$, (3) Hölder's inequality $\|u\|_p \leq \|u\|_2$, and Fubini's Theorem.

Because of an exchange of letters with J. Arazy and J. Peetre, I was led to a generalization of one inequality (i.e. $\int |f|^{p-2}|f'|^2(1-|z|)\,dm \leq \|f\|_p^p$) implicit in Theorem B(ii). This generalization effectively allows 2 to be replaced by numbers s with $2 \leq s < p+2$. §3 contains the proof of this generalization. It is not in any way related to §2 and may be read independently. The proof is not elementary and presumes familiarity with the basic theory of H^p spaces and at least the notion of Carleson measures.

2. Proof of Theorem A. In this section we present our proof of Theorem A. As already mentioned this proof relies on integrating a local estimate. It is this estimate that we establish first.

LEMMA. *If* $2 \leq p < \infty$ *then*

$$|\nabla u(0)|^p \leq 2^{p-1}p(p-1)\int_U |u(z)|^{p-2}|\nabla u(z)|^2 \log\frac{1}{|z|}\,dm(z) \tag{2.1}$$

for every real valued function u *harmonic on* $\overline{U}$.

PROOF. The formula for $u(z)$ in terms of the boundary values of u is

$$u(z) - u(0) = \sum_{n=1}^{\infty} (u_n z^n + u_{-n} \bar{z}^n)$$

where $u_n = \int_0^{2\pi} u(e^{i\theta}) e^{-in\theta}\, d\theta/2\pi$ are the Fourier coefficients of u. Simple calculations show that

$$|\nabla u(0)|^2 = 4|u_1|^2 \le 4(\|u\|_2^2 - |u(0)|^2)$$

by the Parseval relationships. We may, without loss of generality, assume that either $u(0) = 0$ or $u(0) = 1$. In the first case we clearly obtain

$$|\nabla u(0)|^p \le 2^p (\|u\|_p^p - |u(0)|^p) \tag{2.2}$$

because $\|u\|_2 \le \|u\|_p$. In case $u(0) = 1$ we also get (2.2) by applying the easily proved inequality $(x-1)^{p/2} \le x^{p/2} - 1$ for $x \ge 1$ to the right side of the inequality $|\nabla u(0)|^p \le 2^p(\|u\|_p^2 - |u(0)|^2)^{p/2}$ with $x = \|u\|_p^2$ and $|u(0)|^2 = 1$. Combining (2.2) with Theorem B yields the desired conclusion (2.1). □

The part of Theorem B dealing with analytic functions results in a similar inequality for functions f analytic on $\overline{U}$:

$$|f'(0)|^p \le \frac{p^2}{2} \int_U |f'(z)|^2 \log \frac{1}{|z|}\, dm(z). \tag{2.3}$$

The only difference in the proof is the estimate $|f'(0)|^2 \le \|f\|_2^2 - |f(0)|^2$ which loses the factor of 4 present with harmonic functions and their gradients.

We now turn to the proof of Theorem A. Our goal is to establish the inequality

$$\int_U |\nabla u(a)|^p (1-|a|^2)^{p-1}\, dm(a) \le C^p \int_U |u|^{p-2} |\nabla u|^2 \log \frac{1}{|z|}\, dm(z).$$

The Littlewood-Paley Inequality will then follow immediately from Theorem B. We start with inequality (2.1). By rescaling (applying the inequality to $u(z/2)$ and changing variables) we obtain

$$|\nabla u(0)|^p \le \int |u(z)|^{p-2} |\nabla u(z)|^2\, dv(z) \tag{2.4}$$

where v is the measure concentrated on $\{z\colon |z| < 1/2\}$ with

$$dv(z) = 2^{2p-1} p(p-1) \log \frac{1}{2|z|} dm(z),$$

and u is harmonic in U.

Let $Q_a(z) = (a-z)/(1-\bar{a}z)$ so that $u \circ Q_a$ is harmonic in U and apply (2.4) to $u \circ Q_a$ to obtain

$$|\nabla u(a)|^p (1-|a|^2)^p \le \int |u \circ Q_a|^{p-2} |(\nabla u) \circ Q_a|^2 |Q_a'|^2\, dv.$$

Now integrate this inequality with respect to $(1-|a|^2)^{-1} dm(a)$, exhanging integrals on the right, and then estimate the right side in an obvious manner to obtain

$$\begin{aligned} &\int |\nabla u(a)|^p (1-|a|^2)^{p-1}\, dm(a) \\ &\quad \le v(U) \sup_{|z|<1/2} \int |u \circ Q_a(z)|^{p-2} |(\nabla u) \circ Q_a(z)|^2 \frac{|Q_a'(z)|^2}{1-|a|^2}\, dm(a). \end{aligned} \tag{2.5}$$

Thus we are reduced to estimating the supremum in (2.5). Our first step is to change variables by putting $w = Q_a(z)$. The transformation $a \to w$ is not conformal but it is easy to solve for a, obtaining

$$a = [w(1-|z|^2) + z(1-|w|^2)]/(1-|zw|^2).$$

From this and standard mapping properties of Q_a it follows that the transformation $a \to Q_a(z)$ is one-to-one and onto. Computing the Jacobian of this transformation yields

$$dm(a) = \frac{|1-\bar{a}z|^2}{1-|wz|^2} dm(w). \tag{2.6}$$

(The exterior calculus and the formula $dm(w) = (2\pi i)^{-1} d\bar{w} \wedge dw$ help to organize the calculations. Without these it is tedious, but straightforward.) More calculation shows that

$$|Q'_a(z)|^2/(1-|a|^2) = (1-|w|^2)[(1-|z|^2)|1-\bar{a}z|^2]^{-1}.$$

Putting this into (2.6) and (2.5) yields

$$\begin{aligned} &\int |\nabla u(a)|^p (1-|a|^2)^{p-1}\, dm(a) \\ &\quad \le v(U) \sup_{|z|<1/2} \int |u(w)|^{p-2} |\nabla u(w)|^2 \frac{(1-|w|^2) dm(w)}{(1-|z|^2)(1-|wz|^2)} \\ &\quad \le \frac{16}{9} v(U) \int |u(w)|^{p-2} |\nabla u(w)|^2 (1-|w|^2)\, dm(w). \end{aligned} \tag{2.7}$$

Now $1-|w|^2 \le 2\log(1/|w|)$ and $v(U) = 2^{2p-4}p(p-1)$. Put this in (2.7) and combine with Theorem B to obtain

$$\int |\nabla u(a)|^p (1-|a|^2)^{p-1}\, dm(a) \le (2^{2p+2}/9)\|u\|_p^p. \quad \square$$

The calculations for an analytic function go much the same and lead to the desired inquality

$$\int |f'(a)|^p (1-|a|^2)^{p-2}\, dm(a) \le \frac{2^{p+2}}{9}\|f\|_p^p.$$

A more elementary proof with somewhat larger constants can be obtained by making use of the special form that v has. The above approach shows that the inequality (2.4) for *any* finite measure v compactly supported in U suffices to prove Theorem A. (More generally almost any local inequality at the origin will generate some global inequality.)

3. A generalization of Theorem B. In this section we generalize part of Theorem B by proving the following result.

THEOREM C. *Let $0 < p,\ s < +\infty$. Then there exists a constant $C = C(p,s)$ such that*

$$\int_U |f(z)|^{p-s} |f'(z)|^s (1-|z|)^{s-1}\, dm(z) \le C\|f\|_p^p \tag{3.1}$$

for all $f \in H^p$ if and only if $2 \le s < p+2$.

PROOF. If $s \geq p+2$ then $|z|^{p-s}$ is not integrable dm over U so (3.1) fails for $f(z) = z$. If $s < 2$ then (see below) there is a function g belonging to H^s such that $\int |g'|^s(1-|z|)^{s-1}\,dm = +\infty$. Writing $g = h + k$ where h and k are zero free we deduce that such a g exists which is zero free. Put $f = g^{s/p}$ so that $f \in H^p$ and $|g'|^s = (p/s)^s|f|^{p-s}|f'|^s$. Thus (3.1) fails for this f since the left side is infinite while the right is finite.

Now suppose $2 \leq s < p+2$ and let $f \in H^p$. Write $f = Bg$ where B is a Blaschke product and g has no zeros and $\|g\|_p = \|f\|_p$. Now

$$\begin{aligned} |f|^{p-s}|f'|^s &= |Bg|^{p-s}|B'g + Bg'|^s \\ &\leq 2^{s-1}(|g|^p|B|^{p-s}|B'|^s + |B|^p|g|^{p-s}|g'|^s) \end{aligned} \tag{3.2}$$

We may thus estimate the appropriate integral for f in terms of two integrals involving g and B. These estimates follow.

First put $h = g^{p/s}$ so that $h \in H^s$ and $|h'|^s = (p/s)|g|^{p-s}|g'|^s$. Thus

$$\begin{aligned} \int |B|^p|g|^{p-s}|g'|^s(1-|z|)^{s-1}\,dm(z) &\leq C\int |h'|^s(1-|z|)^{s-1}\,dm(z) \\ &\leq C\|h\|_s^s = C\|g\|_p^p = C\|f\|_p^p. \end{aligned} \tag{3.3}$$

by Theorem A.

For the second estimate we apply Carleson's Inequality (see [**1**, pp. 238 and 239]). This states that there is an absolute constant C such that, for every g belonging to H^p, $\int |g|^p\,d\mu \leq C\|\mu\|_*\|g\|_p^p$ where

$$\|\mu\|_* = \sup\left\{\int (1-|a|^2)/|1-\bar{a}z|^2\,d\mu(z) \colon a \in U\right\}.$$

We apply this with $d\mu = |B|^{p-s}|B'|^s(1-|z|^2)^{s-1}dm$. Then we need to estimate $\|\mu\|_* = \sup\{C(a)\colon a \in U\}$ where

$$C(a) = \int \frac{1-|a|^2}{|1-\bar{a}z|^2}|B(z)|^{p-s}|B'(z)|^s(1-|z|^2)^{s-1}\,dm(z).$$

Changing variables by putting $z = (w+a)/(1+\bar{a}w)$ gives

$$C(a) = \int \left|B\left(\frac{w+a}{1+\bar{a}w}\right)\right|^{p-s}\left|\frac{d}{dw}B\left(\frac{w+a}{1+\bar{a}w}\right)\right|^s(1-|w|^2)^{s-1}\,dm(w).$$

We are thus reduced to estimating the left side of (3.1) for certain functions B on the unit sphere of H^∞.

For such B we have, from the conformally invariant form of Schwarz's Lemma, that $|B'(z)|(1-|z|^2) \leq 1-|B(z)|^2 \leq 1$. Thus

$$\int |B(z)|^{p-s}|B'(z)|^s(1-|z|^2)^{s-1}\,dm(z) \leq \int |B(z)|^{q-2}|B'(z)|^2(1-|z|^2)\,dm(z)$$

where $q = p+2-s > 0$. (We have just replaced $|B'(z)|^{s-2}(1-|z|^2)^{s-2}$ by 1, which dominates it.) We conclude that $C(a) \leq C\|B\|_q^p \leq C$ by Theorem B again. Thus

$$\int |g|^p|B|^{p-s}|B'|^s(1-|z|)^{s-1}\,dm \leq C\|g\|_p^p = C\|f\|_p^p.$$

Combining this with (3.3) and (3.2) gives (3.1). □

The statement that there is a function $f \in H^s$ such that $\int |f'|^s(1-|z|)^{s-1}\,dm = +\infty$ whenever $s < 2$ is not easy to come by in print. In [2] there is a construction which yields this result for $1 < s < 2$ and Rudin [5] obtains it for $s = 1$. Both constructions produce a bounded function f. Since bounded functions also satisfy $|f'(z)|(1 - |z|)$ is bounded we see that, for $s < 1$

$$\int |f'(z)|^s(1 - |z|)^{s-1}\,dm \geq c \int |f'(z)|\,dm = +\infty$$

if f is the bounded function of Rudin's example.

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