PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 103, Number 3, July 1988

CLOSED SETS WITHOUT MEASURABLE MATCHING

M. LACZKOVICH

(Communicated by R. Daniel Mauldin)

ABSTRACT. We construct a rectangle in the unit square such that its perimeter contains a matching (i.e. the graph of a bijection of the unit interval onto itself) but does not contain a Borel matching or a matching measurable with respect to the linear measure.

Let I and I^2 denote the unit interval and the unit square, respectively. A subset of I^2 is called a matching if it is the graph of a one-to-one map of I onto itself. By a theorem of D. König [**5**], if each section $K_x = \{y: (x,y) \in K\}$ $(x \in I)$ and $K^y = \{x: (x,y) \in K\}$ $(y \in I)$ of the set $K \subset I^2$ consists of exactly n elements (n is finite), then K contains a matching. For infinite n this was proved by J. Kaniewski and C. A. Rogers [**4**].

It may happen that a Borel subset of I^2 contains a matching but does not contain a Borel matching or even a matching measurable with respect to the linear measure. Borel sets with this property were constructed by Kaniewski and Rogers [**4**] and Mauldin [**6**] (see also [**2**, §4]). The set given in [**4**] has countably infinite sections; the sections of Mauldin's set are uncountable. In this note we present a closed set with the property above. It is the union of four segments and has finite sections.

THEOREM. *Let $u \in I$ be an irrational number and let R denote the perimeter of the rectangle with vertices $A_0(1, 1-u)$, $A_1(1-u, 1)$, $A_2(0, u)$, and $A_3(u, 0)$. Then R contains a matching but does not contain a Borel matching. Moreover, no matching in R is measurable with respect to the linear (Hausdorff) measure.*

PROOF. By symmetry, we may assume $0 < u < 1/2$. The set R defines a bipartite graph G as follows. Let I_1 and I_2 be two copies of I, and let the points $x \in I_1$ and $y \in I_2$ be connected by an edge if and only if $(x,y) \in R$. Then R contains a matching if and only if G contains a 1-factor. Also, G contains a 1-factor if and only if each connected component of G contains a 1-factor. Since every point of G has degree 1 or 2, the connected components of G are paths and circuits. (As for the notions of graph theory used above, see [**1**].) The circuits, infinite paths and finite paths of even length contain 1-factors, and hence, in order to find a 1-factor in G it is enough to prove that no component of G is a finite path of odd length.

Suppose this is not true and let $C = \{x_1, x_2, \ldots, x_n\}$ be a component of G, where $(x_{i-1}, x_i) \in R$ for every $i = 2, \ldots, n$, and n is odd. Then x_1 and x_n both belong to I_1 or I_2 and have degree 1. Since the only elements of degree 1 in G are 0 and 1, this implies that $x_1 = 0$, $x_n = 1$ or $x_1 = 1$, $x_n = 0$.

Received by the editors July 1, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 28A05; Secondary 05C70.

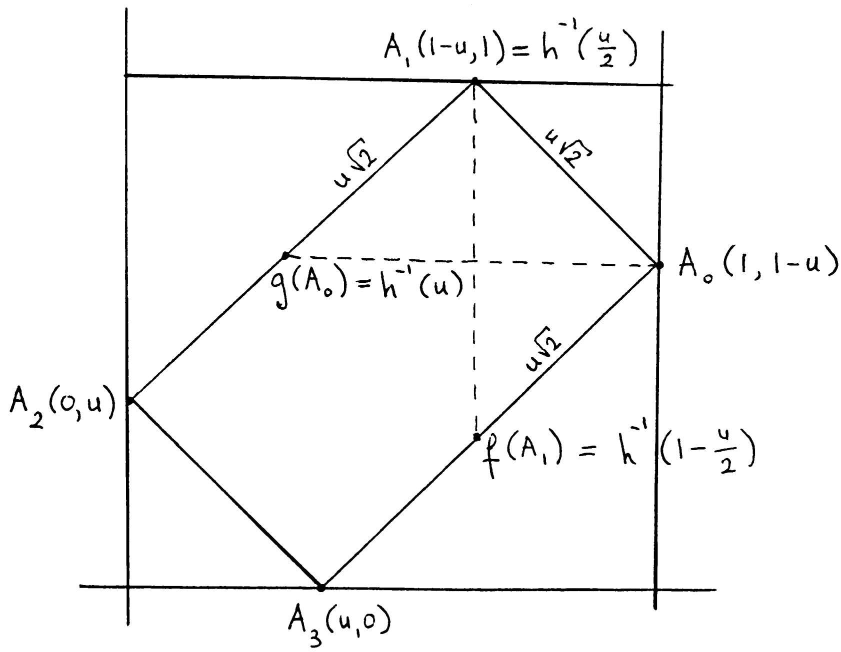

FIGURE 1

If $(x,y) \in R$ and (x,y) belongs to the segment joining A_0 and A_1 then $y = -x+2-u$. If (x,y) belongs to the other three segments then $y = \pm x \pm u$. This implies, by induction on k, that $x_k = \pm x_1 + 2a_k + b_k u$ ($k = 2,\dots,n$), where a_k and b_k are integers. In particular, $x_n = \pm x_1 + 2a_n + b_n u$, and hence $b_n u = x_n \pm x_1 - 2a_n = \pm 1 - 2a_n$. Thus $b_n u$ is an odd integer which implies $b_n \neq 0$. Therefore u is rational, which contradicts our assumption. Hence G does not contain components of odd length and, consequently, contains a matching.

Let μ denote the normalized linear measure on R, i.e. let $\mu(H) = \lambda_1(H)/2\sqrt{2}$ for every measurable $H \subset R$. We define two maps, f and g, of R into itself as follows. We put $f(A_0) = A_0$ and $f(A_2) = A_2$. If $(x,y) \in R$ and $x \neq 0,1$ then there is a unique $z \neq y$ such that $(x,z) \in R$, and we define $f(x,y) = (x,z)$. Also, we define $g(A_1) = A_1$ and $g(A_3) = A_3$. If $(x,y) \in R$ and $y \neq 0,1$ then there is a $z \neq x$ such that $(z,y) \in R$ and we put $g(x,y) = (z,y)$. It is easy to check that f and g are measure-preserving homeomorphisms of R onto itself with $f^{-1} = f$ and $g^{-1} = g$.

We prove that $g \circ f$ is ergodic on R, that is, if H is measurable and $g \circ f(H) = H$ then $\mu(H) = 0$ or $\mu(H) = 1$.

Let T denote the circle group with the Lebesgue measure, and let $h: R \to T$ be a measure-preserving homeomorphism of R onto T such that $h(A_0) = 0$, $h(A_1) = u/2$, $h(A_2) = 1/2$, and $h(A_3) = (1+u)/2$. Let $k = h \circ g \circ f \circ h^{-1}$, then k is a measure-preserving homeomorphism of T onto itself. Therefore either $k(t) = t + c$ or $k(t) = -t + c$ ($t \in T$) with a constant $c \in T$. It is easy to check that $k(0) = u$ and $k(1 - (u/2)) = u/2$ (see Figure 1). Hence $k(t) = t + u$ for every $t \in T$. Since u is irrational, k is ergodic on T (see [**3**, p. 26]), and hence $g \circ f = h^{-1} \circ k \circ h$ is ergodic on R. (This argument is similar to that in [**8**, p. 10].)

Now suppose that H is a measurable matching in R. Then $H \cap f(H)$ is finite and $H \cup f(H) = R$. This implies $\mu(H) = 1/2$. Also, $H \cap g(H)$ is finite and $H \cup g(H) = R$, and hence the symmetric difference of the sets H and $g \circ f(H)$ is finite. Since $g \circ f$ is ergodic, this gives $\mu(H) = 0$ or $\mu(H) = 1$. This contradiction completes the proof.

REMARK. It was proved by R. Rado in [7] that if the sections of a set $K \subset I^2$ are finite then K contains a matching if and only if $\operatorname{card} \bigcup\{K_x : x \in H\} \geq \operatorname{card} H$ and $\operatorname{card} \bigcup\{K^y : y \in H\} \geq \operatorname{card} H$ hold for every finite set $H \subset I$.

Our theorem shows that Rado's condition does not ensure the existence of a Borel matching even if K is compact. The same is true for König's condition. Indeed, let R be a rectangle as in the theorem above, and let K be the union of R and the points $(0,0)$ and $(1,1)$. Then K is compact, $\operatorname{card} K_x = \operatorname{card} K^y = 2$ for every $x, y \in I$ and K does not contain a measurable matching.

REFERENCES

1. B. Bollobás, *Graph theory*, Graduate Texts in Math., vol. 63, Springer-Verlag, Berlin and New York, 1979.
2. S. Graf and R. D. Mauldin, *Measurable one-to-one selections and transition kernels*, Amer. J. Math. **107** (1985), 407–426.
3. P. R. Halmos, *Lectures on ergodic theory*, Chelsea, 1956.
4. J. Kaniewski and C. A. Rogers, *Double uniformization*, J. London Math. Soc. (2) **22** (1980), 521–533.
5. D. König, *Sur les correspondences multivoques des ensembles*, Fund. Math. **8** (1926), 114–134.
6. R. D. Mauldin, *One-to-one selections-marriage theorems*, Amer. J. Math. **104** (1982), 823–828.
7. R. Rado, *Factorization of even graphs*, Quart. J. Math. Oxford Ser. **20** (1949), 95–104.
8. V. N. Sudakov, *On independent complementation to two partitions in the case when there exists a bounded density*, Proc. Steklov Inst. Math. **111** (1970), 3–11 (Russian).

DEPARTMENT OF ANALYSIS, EÖTVÖS LORÁND UNIVERSITY, BUDAPEST, HUNGARY