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SUBNORMAL ELEMENTS OF C^*-ALGEBRAS

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ABSTRACT. An algebraic characterization of a subnormal operator on Hilbert space is given. The result also characterizes subnormal elements of certain abstract C^*-algebras

Throughout this note $\mathcal{H}$ will denote a complex Hilbert space and $\mathcal{L}(\mathcal{H})$ the C^*-algebra of all bounded linear operators on $\mathcal{H}$. Recall that an operator S in $\mathcal{L}(\mathcal{H})$ is said to be subnormal if there exists a Hilbert space $\mathcal{K}$ which contains $\mathcal{H}$ and a normal operator N in $\mathcal{L}(\mathcal{K})$ such that $\mathcal{H}$ is invariant for N and $N|\mathcal{H} = S$. For an element T in a C^*-algebra $\mathcal{A}$, let $C^*(T)$ denote the C^*-algebra generated by T and 1. There are several known characterizations of subnormal operators in the C^*-algebra $\mathcal{L}(\mathcal{H})$. The following theorem appears in [5].

THEOREM A. *The following statements are equivalent for an operator S in $\mathcal{L}(\mathcal{H})$:*

1. *S is subnormal.*
2. *S has a quasinormal extension.*
3. *(Halmos [8]). For any $f_0, f_1, \ldots, f_n$ in $\mathcal{H}$,*

$$\sum_{j=0}^{n}\sum_{k=0}^{n}\langle S^j f_k, S^k f_j\rangle \geq 0,$$

and there exists a constant $c > 0$ such that for any $f_0, f_1, \ldots, f_n$ in $\mathcal{H}$,

$$\sum_{j=0}^{n}\sum_{k=0}^{n}\langle S^{j+1} f_k, S^{k+1} f_j\rangle \leq c\sum_{j=0}^{n}\sum_{k=0}^{n}\langle S^j f_k, S^k f_j\rangle.$$

4. *(Bram [2]). For any $f_0, f_1, \ldots, f_n$ in $\mathcal{H}$,*

$$\sum_{j=0}^{n}\sum_{k=0}^{n}\langle S^j f_k, S^k f_j\rangle \geq 0.$$

5. *(Embry [7]). For any $f_0, f_1, \ldots, f_n$ in $\mathcal{H}$.*

$$\sum_{j=0}^{n}\sum_{k=0}^{n}\langle S^{j+k} f_j, S^{j+k} f_k\rangle \geq 0.$$

6. *(Bunce and Deddens [4]). For any $B_0, B_1, \ldots, B_n$ in $C^*(S)$,*

$$\sum_{j=0}^{n}\sum_{k=0}^{n} B_j^* S^{*k} S^j B_k \geq 0.$$

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7. (*Embry* [7]). *There exists a positive operator-valued measure Q on some interval $[0,a]$, $a > 0$, such that for every integer $n > 0$,*

$$S^{*n}S^n = \int_0^a t^{2n}\,dQ(t).$$

Note that each of the above characterizations of subnormal operators is spatial except for number 6. Bunce used number 6 to define a subnormal element of an abstract C^*-algebra (cf. [3]). An element S of a C^*-algebra $\mathcal{A}$ is said to be *subnormal* if for all $B_0, B_1, \ldots, B_n$ in $\mathcal{A}$, $\sum_{j=0}^n \sum_{k=0}^n B_j^* S^{*k} S^j B_k \geq 0$. (According to Remark 1 of [4], an operator S in $\mathcal{L}(\mathcal{H})$ is subnormal if and only if for any $B_0, B_1, \ldots, B_n$ in $\mathcal{L}(\mathcal{H})$, $\sum_{j=0}^n \sum_{k=0}^n B_j^* S^{*k} S^j B_k \geq 0$. Thus it is correct to allow B_k to be an element of $\mathcal{A}$ and not insist that B_k belongs to $C^*(S)$.) The purpose of this note is to give another characterization of subnormal elements of a C^*-algebra provided that the algebra is "large enough."

We shall begin with the following theorem. Recall that an element W of a C^*-algebra is said to be an *isometry* if $W^*W = 1$.

THEOREM 1. *Suppose that $\mathcal{A}$ is a C^*-algebra and $S \in \mathcal{A}$. Suppose that there exist a normal element N and an isometry W in $\mathcal{A}$ such that $S = W^*NW$ and $NP = PNP$, where $P = WW^*$. Then S is a subnormal element of $\mathcal{A}$.*

PROOF. Let k be a nonnegative integer, and assume that $S^k = W^*N^kW$. Then

$$\begin{aligned} S^{k+1} &= S^kSW^*W = W^*N^kWW^*NWW^*W \\ &= W^*N^kPNPW = W^*N^kNPW = W^*N^{k+1}W. \end{aligned}$$

Hence an induction argument implies that $S^k = W^*N^kW$ for each nonnegative integer k. Thus $S^{*k} = W^*N^{*k}W$ and

$$\begin{aligned} S^{*k}S^j &= S^{*k}S^jW^*W = W^*N^{*k}WW^*N^jWW^*W \\ &= W^*N^{*k}PN^jPW = W^*N^{*k}N^jPW = W^*N^jN^{*k}W \end{aligned}$$

for all nonnegative integers j and k. (We have used the fact that $NP = PNP$ implies that $N^jP = PN^jP$ for each nonnegative integer j.) Thus

$$B_j^*S^{*k}S^jB_k = B_j^*W^*N^jN^{*k}WB_k = (N^{*j}WB_j)^*(N^{*k}WB_k) = E_j^*E_k,$$

$j, k = 0, 1, \ldots, n$, where $E_i = N^{*i}WB_i$, $i = 0, 1, \ldots, n$. It follows that

$$\begin{aligned} \sum_{j=0}^n \sum_{k=0}^n B_j^*S^{*k}S^jB_k &= \sum_{j=0}^n \sum_{k=0}^n E_j^*E_k = \left(\sum_{j=0}^n E_j^*\right)\left(\sum_{k=0}^n E_k\right) \\ &= \left(\sum_{k=0}^n E_k\right)^*\left(\sum_{k=0}^n E_k\right) \geq 0. \end{aligned}$$

Therefore, S is a subnormal element of $\mathcal{A}$.

Note that when $\mathcal{A} = \mathcal{L}(\mathcal{H})$, the condition $NP = PNP$ simply asserts that the range of W is invariant for N.

Is the converse of Theorem 1 true? We shall see that the answer to this question is affirmative if the C^*-algebra is "large enough". The following theorem shows that the answer to the above question is affirmative for $\mathcal{A} = \mathcal{L}(\mathcal{H})$.

We shall call a pair of isometries W_1 and W_2 in a C^*-algebra $\mathcal{A}$ *complementary* if $W_1W_1^* + W_2W_2^* = 1$. The element W_1 is called a *complement* of W_2, and vice versa.

THEOREM 2. *Suppose that S is a subnormal operator in $\mathcal{L}(\mathcal{H})$. Then there exist a normal operator N and an isometry W in $\mathcal{L}(\mathcal{H})$ such that $S = W^*NW$ and $NP = PNP$, where $P = WW^*$.*

PROOF. Since each subnormal operator on a finite dimensional Hilbert space is normal, we may assume that $\mathcal{H}$ is infinite dimensional. Let M be the minimal normal extension of S. Halmos showed in [**8**] (see also [**1**]) that M is unitarily equivalent to the operator

$$\begin{bmatrix} S & X \\ 0 & T^* \end{bmatrix}$$

on $\mathcal{H}\oplus\mathcal{H}$. A routine matrix calculation shows that $S^*S-SS^* = XX^*, T^*T-TT^* = X^*X$, and $XT = S^*X$. Let W_1 and W_2 be a pair of complementary isometries in $\mathcal{L}(\mathcal{H})$. Let $N = W_1SW_1^* + W_1XW_2^* + W_2T^*W_2^*$. Observe that $W_2^*W_1 = W_1^*W_2 = 0$. Now an easy calculation shows that $N^*N - NN^* = 0$. Thus N is normal, and it is straightforward to verify that $W_1^*NW_1 = S$ and $NP = PNP(= W_1SW_1^*)$.

The operator T in the proof of Theorem 2 is called the dual of S (cf. [**6**]). Let $\mathcal{A}$ be an abstract C^*-algebra and let S be a subnormal element of $\mathcal{A}$. We shall call an element T in $\mathcal{A}$ the *algebraic dual* of S if there exists an element X in $\mathcal{A}$ such that $S^*S - SS^* = XX^*$, $T^*T - TT^* = X^*X$, and $XT = S^*X$. (Observe that an algebraic dual of a subnormal operator S in $\mathcal{L}(\mathcal{H})$ is an operator T as in the proof of Theorem 2 for any normal extension (not necessarily the minimal normal extension) of S.) Conway observed that any two duals of a subnormal operator S in $\mathcal{L}(\mathcal{H})$ are unitarily equivalent. It is easy to verify that if T is an algebraic dual of a subnormal element of a C^*-algebra $\mathcal{A}$ and if T_1 is unitarily equivalent to T, then T_1 is also an algebraic dual of S. (Let $X_1 = XU$ where $T_1 = U^*TU$ and $U^*U = UU^* = 1$.) However, two algebraic duals of a subnormal element of a C^*-algebra need not be unitarily equivalent. For example, let $\mathcal{A} = \mathcal{L}(\mathcal{H})$ for a separable, infinite dimensional Hilbert space $\mathcal{H}$, and let V be the unilateral shift in $\mathcal{L}(\mathcal{H})$. Then the dual of V is V itself and $X = 1 - VV^*$. Then for any normal element N in $\mathcal{L}(\mathcal{H})$, the operator

$$\begin{bmatrix} V & X & 0 \\ 0 & V^* & 0 \\ 0 & 0 & N^* \end{bmatrix}$$

is a normal extension of V. Thus $V \oplus N$ is unitarily equivalent to an algebraic dual of V.

We shall say that a C^*-algebra $\mathcal{A}$ is *dual-closed* if for each subnormal element S of $\mathcal{A}$, there exists an element T of $\mathcal{A}$ such that T is an algebraic dual of S. We have the following result for an abstract C^*-algebra.

THEOREM 3. *Suppose that $\mathcal{A}$ is a C^*-algebra that has a complementary pair of isometries and is dual closed, and suppose that S belongs to $\mathcal{A}$. Then S is subnormal if and only if there exist a normal N and an isometry W in $\mathcal{A}$ such that $S = W^*NW$ and $NP = PNP$, where $P = WW^*$. Furthermore, if W is any isometry in $\mathcal{A}$ that* *has a complement in $\mathcal{A}$ and if S is a subnormal element of A, then there exists a normal element N of $\mathcal{A}$ such that $S = W^*NW$ and $NP = PNP$, where $P = WW^*$.*

PROOF. Consider the first assertion in the statement of the theorem. Since the proof of Theorem 2 is all algebraic after the point where we assert the existence of the operators X and T, the stated necessary conditions for the subnormality of S follow from the same proof. The fact that the stated conditions are sufficient for the subnormality of S is just Theorem 1. The second assertion in the statement of the theorem also follows from the proof of Theorem 2.

The key ingredient in proving that the conditions that are stated in Theorem 3 are necessary conditions for the subnormality of S is the fact that the algebra $\mathcal{A}$ contains the algebraic dual of S. Accordingly, let $\mathcal{A}$ be a C^*-algebra that contains a pair of complementary isometries and that is *not* necessarily dual-closed. Then if S is a subnormal element of $\mathcal{A}$ and if $\mathcal{A}$ contains an algebraic dual of S, then there exist a normal N and an isometry W in $\mathcal{A}$ such that $S = W^*NW$ and $NP = PNP$, where $P = WW^*$.

Observe that if $\mathcal{A}$ is $\mathcal{L}(\mathcal{H})$ for an infinite dimensional Hilbert space $\mathcal{H}$ or $\mathcal{A}$ is the Calkin algebra, then $\mathcal{A}$ contains a pair of complementary isometries. However, there exist C^*-algebras which do not contain a pair of complementary isometries. For example, if $\mathcal{A}$ is a commutative algebra, then $\mathcal{A}$ does not contain a pair of complementary isometries. (In this case $\mathcal{A}$ does not even contain any nonunitary isometries.) Observe that $\mathcal{L}(\mathcal{H})$ is dual-closed.

QUESTION. Is the Calkin algebra dual-closed?

Let $\mathcal{A}$ be a C^*-algebra, S a subnormal element of $\mathcal{A}$, and T an algebraic dual of S. Since $\mathcal{A}$ has faithful representation in $\mathcal{L}(\mathcal{K})$, for some Hilbert space $\mathcal{K}$, the element T is also subnormal. In the following we provide an algebraic proof of this fact under certain conditions on $\mathcal{A}$.

THEOREM 4. *Suppose that $\mathcal{A}$ is a C^*-algebra and $\mathcal{A}$ has a complementary pair of isometries. Suppose that S is a subnormal element of $\mathcal{A}$. If T belongs to $\mathcal{A}$ and T is an algebraic dual of S, then T is subnormal.*

PROOF. In the proof of Theorem 2, it is easy to verify that $T = W_2^*N^*W_2$ and $N^*E = EN^*E$, where $E = W_2W_2^*$. Hence T is subnormal by Theorem 1.

THEOREM 5. *Suppose that S belongs to $\mathcal{L}(\mathcal{H})$ and there exist a normal operator N and an isometry W in $\mathcal{L}(\mathcal{H})$ such that $S = W^*NW$ and $NP = PNP$, where $P = WW^*$. If the orthocomplement of the range of W is finite dimensional, then S is normal.*

PROOF. The equation $NP = PNP$ implies that the range of W, $\mathcal{R}(W)$, is invariant for N. Thus $\mathcal{R}(W)^\perp$ is invariant for N^*.

Since $N^*|\mathcal{R}(W)^\perp$ is subnormal and $\mathcal{R}(W)^\perp$ is finite dimensional, then $N^*|\mathcal{R}(W)^\perp$ is normal. It follows that both $\mathcal{R}(W)^\perp$ and $\mathcal{R}(W)$ reduce N. Thus $NP = PN$, and a calculation shows that $S^*S - SS^* = W^*(N^*N - NN^*)W = 0$. Therefore, S is normal.

We have seen that C^*-algebras that have a pair of complementary isometries and that are dual-closed are "large enough" for the converse of Theorem 1 to hold. The following example shows that the converse of Theorem 1 can fail if the C^*-algebra is not "large enough". For T in $\mathcal{L}(\mathcal{H})$, let $\tilde{T}$ denote the image of T in the Calkin algebra under the natural quotient map of $\mathcal{L}(\mathcal{H})$ onto $\mathcal{L}(\mathcal{H})/\mathcal{C}$, where $\mathcal{C}$ denotes the ideal of all compact operators in $\mathcal{L}(\mathcal{H})$.

EXAMPLE 6. Let V denote the unilateral shift of multiplicity 1 in $\mathcal{L}(\mathcal{H})$, where $\mathcal{H}$ is separable and infinite dimensional. Let $\mathcal{A} = C^*(V)$. Then $\mathcal{A}$ does not have a pair of complementary isometries. (Indeed, if W_1 and W_2 are isometries in $\mathcal{A}$ and $W_1W_1^* + W_2W_2^* = 1$, then $\tilde{W}_1$ and $\tilde{W}_2$ are isometries in $\mathcal{B} = C^*(\tilde{V})$. But $\mathcal{B}$ is a commutative algebra since it is generated by $\tilde{V}$, which is normal, and 1. Thus, $1 = \tilde{W}_1\tilde{W}_1^* + \tilde{W}_2\tilde{W}_2^* = \tilde{W}_1^*\tilde{W}_1 + \tilde{W}_2^*\tilde{W}_2 = 1 + 1 = 2$.) In fact, if W is an isometry in $\mathcal{A}$, then $\tilde{W}\tilde{W}^* = \tilde{W}^*\tilde{W} = 1$. Thus $\mathcal{R}(W)^{\perp}$ is finite dimensional, and Theorem 6 implies W^*NW is normal for each normal operator N in $\mathcal{A}$. Thus, the converse of Theorem 1 does not hold for $\mathcal{A} = C^*(V)$ and $S = V$. Observe that there exist a normal operator N and an isometry W in $\mathcal{L}(\mathcal{H})$ such that $V = W^*NW$ and $NP = PNP$, where $P = WW^*$. (But of course W is not in $C^*(V)$.) Thus the algebra is just not large enough to contain both N and W.

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