TOPOLOGICAL INVARIANCE OF WEIGHTS FOR WEIGHTED HOMOGENEOUS ISOLATED SINGULARITIES IN $C^3$

OSAMU SAEKI

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ABSTRACT. We show that the weights of a weighted homogeneous polynomial $f$ in $C^3$ with an isolated singularity are local topological invariants of $(C^3, f^{-1}(0))$ at the origin.

1. Introduction. A polynomial $f(z_1, \ldots, z_n)$ in $C^n$ is called weighted homogeneous if there exist positive rational numbers $(w_1, \ldots, w_n)$ such that for every monomial $cz_1^{a_1} \cdots z_n^{a_n}$ ($c \neq 0$) of $f$, $\sum_{i=1}^{n} a_i/w_i = 1$. We call $(w_1, \ldots, w_n)$ the weights of $f$. If $\{z \in C^n; \partial f/\partial z_1(z) = \cdots = \partial f/\partial z_n(z) = 0\} = \{0\}$ as germs at the origin in $C^n$, we say that $f$ has an isolated singularity at the origin. In this case we may assume $w_i \geq 2$ after a suitable coordinate transformation [16]. Thus in this paper we assume that all weights are greater than or equal to 2.

Saito [16] proved that weights are local analytic invariants of $(C^n, f^{-1}(0))$ at the origin for every weighted homogeneous polynomial $f$ with an isolated singularity. Then it arises the question whether they are topological invariants or not. Yoshinaga-Suzuki [18] have answered the question affirmatively in the case $n = 2$. (See also [7].) In this paper we answer the question affirmatively in the case $n = 3$, namely

THEOREM 1. Let $f(z, y, z)$ (resp. $g(x, y, z)$) be a weighted homogeneous polynomial with weights $(w_1, w_2, w_3)$ (resp. $(w'_1, w'_2, w'_3)$) with an isolated singularity. If $(C^3, f^{-1}(0))$ is locally homeomorphic to $(C^3, g^{-1}(0))$ at the origin, then we have $w_i = w'_i$ ($i = 1, 2, 3$) up to order.

REMARK. Let $f$ be a weighted homogeneous polynomial in $C^n$ with an isolated singularity at the origin. Then the local topological type of $(C^n, f^{-1}(0))$ at the origin is determined by the weights of $f$ [8].

Our proof of Theorem 1 heavily depends on a result of Orlik-Vogt-Zieschang [13] about Seifert 3-manifolds and a result of Yoshinaga [17] about characteristic polynomials. As a corollary to Theorem 1 we deduce the case $n = 2$, using the cyclic suspension of knots. Furthermore we consider an application to Zariski’s question [19] concerning the multiplicity of a hypersurface singularity (§3).

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2. Proof of Theorem 1. Let $f$ be a polynomial in $C^n$ ($f(0) = 0$) with an isolated singularity at the origin. Then we define the $(2n-1)$-manifold $K_f$ by $K_f = S^{2n-1}_\varepsilon \cap f^{-1}(0)$, where $S^{2n-1}_\varepsilon$ is the $(2n-1)$-sphere of radius $\varepsilon$ centered at

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the origin and \( \varepsilon > 0 \) is sufficiently small [4]. We call \( K_f \) the link of the singularity of \( f \). Furthermore we denote by \( \Delta_f(t) \) the characteristic polynomial of the Milnor fibration of \( f \) [4]. Then one has the following.

**Lemma 2.** Let \( f \) and \( g \) be polynomials in \( \mathbf{C}^n \) \((f(0) = g(0) = 0)\) with isolated singularities at the origin. If \((\mathbf{C}^n, f^{-1}(0))\) is locally homeomorphic to \((\mathbf{C}^n, g^{-1}(0))\) at the origin, then we have

(i) \( \pi_1(K_f) \cong \pi_1(K_g) \) and

(ii) \( \Delta_f(t) = \Delta_g(t) \).

Part (i) of the above lemma is proved by the same argument as in [14, §2]. Part (ii) is a result of Lê [3].

For the proof of Theorem 1, it suffices to prove the following.

**Theorem 3.** Let \( f(x, y, z) \) (resp. \( g(x, y, z) \)) be a weighted homogeneous polynomial with weights \((w_1, w_2, w_3)\) (resp. \((w'_1, w'_2, w'_3)\)) with an isolated singularity. If \( \pi_1(K_f) \cong \pi_1(K_g) \) and \( \Delta_f(t) = \Delta_g(t) \), then we have \( w_i = w'_i \) \((i = 1, 2, 3)\) up to order.

**Proof.** First, we assume that \( \pi_1(K_f) \) is a finite group. Then \( \pi_1(K_g) \) is also finite. In this case \( f \) and \( g \) have simple critical points at the origin [2]. Hence they are right equivalent to one of the five classes of germs in Table 1 of [2]. Since the five classes of germs are weighted homogeneous with distinct local fundamental groups and weights are invariant under a coordinate transformation [16], \( f \) and \( g \) have the same weights. In case \( \pi_1(K_f) \) is infinite nilpotent, a similar argument can be used. Thus in the following we assume that \( \pi_1(K_f) \cong \pi_1(K_g) \) is neither finite nor infinite nilpotent.

Remember that \( K_f \) and \( K_g \) admit natural fixed point free actions of \( S^1 \) [11]. In other words, they are Seifert 3-manifolds. Since \( \pi_1(K_f) \cong \pi_1(K_g) \) is neither finite nor infinite nilpotent, \( K_f \) and \( K_g \) are large Seifert 3-manifolds in the sense of Orlik [10]. (Note that the small Seifert 3-manifold with Seifert invariants \((-2; (0, 0, 0), (2, 1), (2, 1), (2, 1), (2, 1))\) does not occur as the link of an isolated singularity, since its Euler number is zero. See [6].) Thus \( K_f \) and \( K_g \) have the same Seifert invariants by Orlik-Vogt-Zieschang [13, 10, p. 97]. (Note that \( K_f \) and \( K_g \) are orientation preservingly homeomorphic to each other.)

Now let \( w_i = u_i/v_i \) and \( w'_i = u'_i/v'_i \) be irreducible fractions. Then by Yoshinaga [17], the following (1) and (2) hold.

(1) \( \{2, u_1, u_2, u_3\} = \{2, u'_1, u'_2, u'_3\} \).

(2) For every \( u \in \{2, u_1, u_2, u_3\} \),

\[
\prod_{u_i = u} \left(1 - \frac{u_i}{v_i}\right) = \prod_{u'_i = u} \left(1 - \frac{u'_i}{v'_i}\right)
\]

where the product over an empty set is assumed to be one.

**Case 1.** \( u_1, u_2, \) and \( u_3 \) are pairwise distinct.

Using (1) and (2), we see easily \( u_i/v_i = u'_i/v'_i \) up to order.

**Case 2.** \( u_1 = u_2 = u_3 \).

If \( u_i = 2 \), \( \pi_1(K_f) \) is finite. Thus we may assume \( u_i > 2 \). Then we have \( u'_1 = u'_2 = u'_3 \) or \( u'_1 = u'_2 \neq u'_3 = 2 \) or \( u'_1 \neq u'_2 = u'_3 = 2 \) by (1). In case
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\[ u'_1 = u'_2 \neq u'_3 = 2, \]

\[ \prod_{u_i = 2} \left( 1 - \frac{u_i}{v_i} \right) \neq \prod_{u'_i = 2} \left( 1 - \frac{u'_i}{v'_i} \right). \]

This contradicts (2). In case \( u'_1 \neq u'_2 = u'_3 = 2 \), \( \pi_1(K_g) \) is finite. Thus we may assume \( u_1 = u_2 = u_3 = u'_1 = u'_2 = u'_3. \)

By [11 and 12] (see also [15]), we see that the Seifert 3-manifold \( K_f \) (resp. \( K_g \)) has stability groups of orders \( v_1, v_2 \), and \( v_3 \) (resp. \( v'_1, v'_2 \), and \( v'_3 \)). (Possibly \( v_i = 1 \) or \( v'_i = 1 \).) Since \( K_f \) and \( K_g \) have the same Seifert invariants, we have \( v_i = v'_i \) up to order. Hence \( u_i/v_i = u'_i/v'_i \).

**Case 3.** \( u_1 \neq u_2 = u_3. \)

Using (1) and (2), we obtain \( u_1 = u'_1 \neq u_2 = u_3 = u'_2 = u'_3 \) after renumbering indices if necessary. Suppose that the Seifert 3-manifold \( K_f \) (resp. \( K_g \)) has Seifert invariants \( \{-b; (a, h, 0, 0); (\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r)\} \) (resp. \( \{-b'; (a, h', 0, 0); (\alpha'_1, \beta'_1), \ldots, (\alpha'_r, \beta'_r)\} \)). By [11], \( b = d/q_1q_2q_3 + \sum_{i=1}^r \beta_i/\alpha_i \) and \( b' = d'/q'_1q'_2q'_3 + \sum_{i=1}^r \beta'_i/\alpha'_i \), where \( d \) (resp. \( d' \)) is the least common multiple of \( u_1, u_2, \) and \( u_3 \) (resp. \( u'_1, u'_2, \) and \( u'_3 \)) and \( q_i \) (resp. \( q'_i \)) is \( d v_i/u_i \) (resp. \( d' v'_i/u'_i \)). Since \( d = d', b = b', r = s, \) and \( (\alpha_i, \beta_i) = (\alpha'_i, \beta'_i) \) up to order, we have \( q_1q_2q_3 = q'_1q'_2q'_3 \). Thus we obtain \( v_1v_2v_3 = v'_1v'_2v'_3 \). Using this equation and (2), we see \( v_1 = v'_1, v_2v_3 = v'_2v'_3 \), and \( v_2 + v_3 = v'_2 + v'_3 \). Thus we obtain \( v_2 = v'_2 \) and \( v_3 = v'_3 \) after renumbering indices if necessary. Hence \( v_i = v'_i \). This completes the proof.

**REMARK.** (1) Orlik [9] asserts that, for polynomials \( f \) and \( g \) as in Theorem 1, if \( K_f \) and \( K_g \) are not lens spaces and \( \pi_1(K_f) \) is isomorphic to \( \pi_1(K_g) \), then \( f \) and \( g \) have the same weights. If this assertion were true, Theorem 1 follows immediately. However, the above assertion of Orlik is not true. For example, \( x^2 + y^7 + z^{14\alpha} \) and \( x^3 + y^4 + z^{12\alpha} \) (\( \alpha \geq 1 \)) have homeomorphic links which are not lens spaces; however, they have distinct weights. In fact, their links are homeomorphic to the Seifert 3-manifold with Seifert invariants \( \{-1; (a, 3, 0, 0); (a, a - 1)\} \) (see [10, 11]). Note that the results of Orlik [9] remain true, since he uses the above assertion only in the case that the fundamental groups are finite or infinite nilpotent.

(2) In higher dimensions, Theorem 3 does not hold. For example,

\[ f(z_1, \ldots, z_n) = z_1^2z_2 + z_1z_2^6 + z_3^3 + z_4^{13} + z_5^2 + \cdots + z_n^2 \]

and

\[ g(z_1, \ldots, z_n) = z_1^3z_2 + z_1z_2^4 + z_3^3 + z_4^{13} + z_5^2 + \cdots + z_n^2 \quad (n \geq 4) \]

have homeomorphic links and the same characteristic polynomial; however, they have distinct weights. In fact, \( (C^n, f^{-1}(0)) \) is not locally homeomorphic to \( (C^n, g^{-1}(0)) \) at the origin [15]. We do not know whether Theorem 1 also holds in higher dimensions.

**3. Applications.** As a corollary to Theorem 1, we have the following result concerning the case \( n = 2 \).

**Corollary 4 (Yoshinaga-Suzuki [18], Nishimura [7]).** Let \( f(x,y) \) (resp. \( g(x,y) \)) be a weighted homogeneous polynomial with weights \( (w_1, w_2) \) (resp. \( (w'_1, w'_2) \)) with an isolated singularity. If \( (C^2, f^{-1}(0)) \) is locally homeomorphic to \( (C^2, g^{-1}(0)) \) at the origin, then we have \( w_i = w'_i \) (\( i = 1, 2 \)) up to order.

To prove Corollary 4, we need the following.
LEMMA 5. Let \( f \) and \( g \) be polynomials in \( \mathbb{C}^2 \) \((f(0) = g(0) = 0)\) with isolated singularities at the origin. If \((\mathbb{C}^2, f^{-1}(0))\) is locally homeomorphic to \((\mathbb{C}^2, g^{-1}(0))\) at the origin, then \((S^3_e, K_f)\) is relatively homeomorphic to \((S^3_e, K_g)\).

One can prove Lemma 5 using the following two facts and Lemma 2 (ii). The detailed proof is omitted.

FACT 1 [1]. If \( f \) and \( g \) are irreducible polynomials and \( \Delta_f(t) = \Delta_g(t) \), then \((S^3_e, K_f)\) is relatively homeomorphic to \((S^3_e, K_g)\).

FACT 2 [20]. Let \( f = p_1 \cdots p_r \) and \( g = q_1 \cdots q_r \) be irreducible factorizations. If \((S^3_e, K_{p_i})\) is relatively homeomorphic to \((S^3_e, K_{q_i})\) \((1 \leq i \leq r)\) and the linking number of \( K_{p_i} \) and \( K_{p_j} \) in \( S^3 \) is the same as that of \( K_{q_i} \) and \( K_{q_j} \) \((1 \leq i, j \leq r, i \neq j)\), then \((S^3_e, K_f)\) is relatively homeomorphic to \((S^3_e, K_g)\).

PROOF OF COROLLARY 4. Set \( f_1(x, y, z) = f(x, y) + z^2 \) and \( g_1(x, y, z) = g(x, y) + z^2 \). Then \( f_1 \) (resp. \( g_1 \)) is weighted homogeneous with weights \((w_1, w_2, 2)\) (resp. \((w'_1, w'_2, 2)\)) with an isolated singularity. Note that \((S^5, K_{f_1})\) (resp. \((S^5, K_{g_1})\)) is the 2-fold cyclic suspension of \((S^3_e, K_f)\) (resp. \((S^3_e, K_g)\)) [5]. Since \((S^3_e, K_f)\) is relatively homeomorphic to \((S^3_e, K_g)\) by Lemma 5, \((S^5, K_{f_1})\) is relatively homeomorphic to \((S^5, K_{g_1})\). Since \((C^3, f_1^{-1}(0))\) (resp. \((C^3, g_1^{-1}(0))\)) is locally the cone over \((S^3_e, K_{f_1})\) (resp. \((S^3_e, K_{g_1})\)) [4], \((C^3, f_1^{-1}(0))\) is locally homeomorphic to \((C^3, g_1^{-1}(0))\) at the origin. Thus by Theorem 1, \( f_1 \) and \( g_1 \) have the same weights. This implies \( w_i = w'_i \) \((i = 1, 2)\) up to order.

Next we consider an application to Zariski’s question [19]: Is the multiplicity of a hypersurface singularity a topological invariant?

LEMMA 6. Let \( f(z_1, \ldots, z_n) \) be a weighted homogeneous polynomial with weights \((w_1, \ldots, w_n)\) with an isolated singularity. Then the multiplicity of \( f \) at the origin is equal to \( \min\{m \in \mathbb{Z}; m \geq w\} \), where \( w = \min\{w_1, \ldots, w_n\} \).

Lemma 6 follows from the definition of weights and Korollar 1.6 of [16]. The detailed proof is omitted.

Using Lemma 6 and Theorem 1, we have the following immediately.

PROPOSITION 7. Let \( f(x, y, z) \) (resp. \( g(x, y, z) \)) be a weighted homogeneous polynomial in \( \mathbb{C}^3 \) with an isolated singularity. If \((\mathbb{C}^3, f^{-1}(0))\) is locally homeomorphic to \((\mathbb{C}^3, g^{-1}(0))\) at the origin, then \( f \) and \( g \) have the same multiplicity at the origin.

REFERENCES