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TOPOLOGICAL INVARIANCE OF WEIGHTS FOR WEIGHTED HOMOGENEOUS ISOLATED SINGULARITIES IN $\mathbf{C}^3$

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ABSTRACT. We show that the weights of a weighted homogeneous polynomial f in $\mathbf{C}^3$ with an isolated singularity are local topological invariants of $(\mathbf{C}^3, f^{-1}(0))$ at the origin.

1. Introduction. A polynomial $f(z_1, \ldots, z_n)$ in $\mathbf{C}^n$ is called *weighted homogeneous* if there exist positive rational numbers $(w_1, \ldots, w_n)$ such that for every monomial $cz_1^{a_1} \cdots z_n^{a_n}$ $(c \neq 0)$ of f, $\sum_{i=1}^n a_i/w_i = 1$. We call $(w_1, \ldots, w_n)$ the *weights* of f. If $\{z \in \mathbf{C}^n; \partial f/\partial z_1(z) = \cdots = \partial f/\partial z_n(z) = 0\} = \{0\}$ as germs at the origin in $\mathbf{C}^n$, we say that f has an *isolated singularity* at the origin. In this case we may assume $w_i \geq 2$ after a suitable coordinate transformation [**16**]. Thus in this paper we assume that all weights are greater than or equal to 2.

Saito [**16**] proved that weights are local analytic invariants of $(\mathbf{C}^n, f^{-1}(0))$ at the origin for every weighted homogeneous polynomial f with an isolated singularity. Then it arises the question whether they are topological invariants or not. Yoshinaga-Suzuki [**18**] have answered the question affirmatively in the case $n = 2$. (See also [**7**].) In this paper we answer the question affirmatively in the case $n = 3$, namely

THEOREM 1. *Let $f(x, y, z)$ (resp. $g(x, y, z)$) be a weighted homogeneous polynomial with weights (w_1, w_2, w_3) (resp. (w_1', w_2', w_3')) with an isolated singularity. If $(\mathbf{C}^3, f^{-1}(0))$ is locally homeomorphic to $(\mathbf{C}^3, g^{-1}(0))$ at the origin, then we have $w_i = w_i'$ $(i = 1, 2, 3)$ up to order.*

REMARK. Let f be a weighted homogeneous polynomial in $\mathbf{C}^n$ with an isolated singularity at the origin. Then the local topological type of $(\mathbf{C}^n, f^{-1}(0))$ at the origin is determined by the weights of f [**8**].

Our proof of Theorem 1 heavily depends on a result of Orlik-Vogt-Zieschang [**13**] about Seifert 3-manifolds and a result of Yoshinaga [**17**] about characteristic polynomials. As a corollary to Theorem 1 we deduce the case $n = 2$, using the cyclic suspension of knots. Furthermore we consider an application to Zariski's question [**19**] concerning the multiplicity of a hypersurface singularity (§3).

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2. Proof of Theorem 1. Let f be a polynomial in $\mathbf{C}^n$ $(f(0) = 0)$ with an isolated singularity at the origin. Then we define the $(2n-1)$-manifold K_f by $K_f = S_\varepsilon^{2n-1} \cap f^{-1}(0)$, where S_ε^{2n-1} is the $(2n-1)$-sphere of radius ε centered at

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the origin and $\varepsilon > 0$ is sufficiently small [**4**]. We call K_f the *link* of the singularity of f. Furthermore we denote by $\Delta_f(t)$ the characteristic polynomial of the Milnor fibration of f [**4**]. Then one has the following.

LEMMA 2. *Let f and g be polynomials in $\mathbf{C}^n$ ($f(0) = g(0) = 0$) with isolated singularities at the origin. If $(\mathbf{C}^n, f^{-1}(0))$ is locally homeomorphic to $(\mathbf{C}^n, g^{-1}(0))$ at the origin, then we have*
(i) $\pi_1(K_f) \cong \pi_1(K_g)$ *and*
(ii) $\Delta_f(t) = \Delta_g(t)$.

Part (i) of the above lemma is proved by the same argument as in [**14**, §2]. Part (ii) is a result of Lê [**3**].

For the proof of Theorem 1, it suffices to prove the following.

THEOREM 3. *Let $f(x, y, z)$ (resp. $g(x, y, z)$) be a weighted homogeneous polynomial with weights (w_1, w_2, w_3) (resp. (w'_1, w'_2, w'_3)) with an isolated singularity. If $\pi_1(K_f) \cong \pi_1(K_g)$ and $\Delta_f(t) = \Delta_g(t)$, then we have $w_i = w'_i$ ($i = 1, 2, 3$) up to order.*

PROOF. First, we assume that $\pi_1(K_f)$ is a finite group. Then $\pi_1(K_g)$ is also finite. In this case f and g have simple critical points at the origin [**2**]. Hence they are right equivalent to one of the five classes of germs in Table 1 of [**2**]. Since the five classes of germs are weighted homogeneous with distinct local fundamental groups and weights are invariant under a coordinate transformation [**16**], f and g have the same weights. In case $\pi_1(K_f)$ is infinite nilpotent, a similar argument can be used. Thus in the following we assume that $\pi_1(K_f) \cong \pi_1(K_g)$ is neither finite nor infinite nilpotent.

Remember that K_f and K_g admit natural fixed point free actions of S^1 [**11**]. In other words, they are Seifert 3-manifolds. Since $\pi_1(K_f) \cong \pi_1(K_g)$ is neither finite nor infinite nilpotent, K_f and K_g are large Seifert 3-manifolds in the sense of Orlik [**10**]. (Note that the small Seifert 3-manifold with Seifert invariants $\{-2; (o, 0, 0, 0); (2, 1), (2, 1), (2, 1), (2, 1)\}$ does not occur as the link of an isolated singularity, since its Euler number is zero. See [**6**].) Thus K_f and K_g have the same Seifert invariants by Orlik-Vogt-Zieschang [**13**, **10**, p. 97]. (Note that K_f and K_g are *orientation preservingly* homeomorphic to each other.)

Now let $w_i = u_i/v_i$ and $w'_i = u'_i/v'_i$ be irreducible fractions. Then by Yoshinaga [**17**], the following (1) and (2) hold.
(1) $\{2, u_1, u_2, u_3\} = \{2, u'_1, u'_2, u'_3\}$.
(2) For every $u \in \{2, u_1, u_2, u_3\}$,

$$\prod_{u_i = u} \left(1 - \frac{u_i}{v_i}\right) = \prod_{u'_i = u} \left(1 - \frac{u'_i}{v'_i}\right)$$

where the product over an empty set is assumed to be one.

Case 1. *u_1, u_2, and u_3 are pairwise distinct.*

Using (1) and (2), we see easily $u_i/v_i = u'_i/v'_i$ up to order.

Case 2. $u_1 = u_2 = u_3$.

If $u_i = 2$, $\pi_1(K_f)$ is finite. Thus we may assume $u_i > 2$. Then we have $u'_1 = u'_2 = u'_3$ or $u'_1 = u'_2 \neq u'_3 = 2$ or $u'_1 \neq u'_2 = u'_3 = 2$ by (1). In case

$u'_1 = u'_2 \neq u'_3 = 2$,

$$\prod_{u_i=2}\left(1-\frac{u_i}{v_i}\right) \neq \prod_{u'_i=2}\left(1-\frac{u'_i}{v'_i}\right).$$

This contradicts (2). In case $u'_1 \neq u'_2 = u'_3 = 2$, $\pi_1(K_g)$ is finite. Thus we may assume $u_1 = u_2 = u_3 = u'_1 = u'_2 = u'_3$.

By [**11** and **12**] (see also [**15**]), we see that the Seifert 3-manifold K_f (resp. K_g) has stability groups of orders v_1, v_2, and v_3 (resp. v'_1, v'_2, and v'_3). (Possibly $v_i = 1$ or $v'_i = 1$.) Since K_f and K_g have the same Seifert invariants, we have $v_i = v'_i$ up to order. Hence $u_i/v_i = u'_i/v'_i$.

Case 3. $u_1 \neq u_2 = u_3$.

Using (1) and (2), we obtain $u_1 = u'_1 \neq u_2 = u_3 = u'_2 = u'_3$ after renumbering indices if necessary. Suppose that the Seifert 3-manifold K_f (resp. K_g) has Seifert invariants $\{-b; (o,h,0,0); (\alpha_1,\beta_1),\dots,(\alpha_r,\beta_r)\}$ (resp. $\{-b'; (o,h',0,0); (\alpha'_1,\beta'_1),\dots,(\alpha'_s,\beta'_s)\}$). By [**11**], $b = d/q_1q_2q_3 + \sum_{i=1}^{r}\beta_i/\alpha_i$ and $b' = d'/q'_1q'_2q'_3 + \sum_{i=1}^{s}\beta'_i/\alpha'_i$, where d (resp. d') is the least common multiple of u_1, u_2, and u_3 (resp. u'_1, u'_2, and u'_3) and q_i (resp. q'_i) is dv_i/u_i (resp. $d'v'_i/u'_i$). Since $d = d', b = b', r = s$, and $(\alpha_i,\beta_i) = (\alpha'_i,\beta'_i)$ up to order, we have $q_1q_2q_3 = q'_1q'_2q'_3$. Thus we obtain $v_1v_2v_3 = v'_1v'_2v'_3$. Using this equation and (2), we see $v_1 = v'_1$, $v_2v_3 = v'_2v'_3$, and $v_2 + v_3 = v'_2 + v'_3$. Thus we obtain $v_2 = v'_2$ and $v_3 = v'_3$ after renumbering indices if necessary. Hence $w_i = w'_i$. This completes the proof.

REMARK. (1) Orlik [**9**] asserts that, for polynomials f and g as in Theorem 1, if K_f and K_g are not lens spaces and $\pi_1(K_f)$ is isomorphic to $\pi_1(K_g)$, then f and g have the same weights. If this assertion were true, Theorem 1 follows immediately. However, the above assertion of Orlik is not true. For example, $x^2 + y^7 + z^{14\alpha}$ and $x^3 + y^4 + z^{12\alpha}$ $(\alpha \geq 1)$ have homeomorphic links which are not lens spaces; however, they have distinct weights. In fact, their links are homeomorphic to the Seifert 3-manifold with Seifert invariants $\{-1; (o,3,0,0); (\alpha,\alpha-1)\}$ (see [**10, 11**]). Note that the results of Orlik [**9**] remain true, since he uses the above assertion only in the case that the fundamental groups are finite or infinite nilpotent.

(2) In higher dimensions, Theorem 3 does not hold. For example,

$$f(z_1,\dots,z_n) = z_1^2z_2 + z_1z_2^6 + z_3^3 + z_4^{13} + z_5^2 + \cdots + z_n^2$$

and

$$g(z_1,\dots,z_n) = z_1^3z_2 + z_1z_2^4 + z_3^3 + z_4^{13} + z_5^2 + \cdots + z_n^2 \qquad (n \geq 4)$$

have homeomorphic links and the same characteristic polynomial; however, they have distinct weights. In fact, $(\mathbf{C}^n, f^{-1}(0))$ is not locally homeomorphic to $(\mathbf{C}^n, g^{-1}(0))$ at the origin [**15**]. We do not know whether Theorem 1 also holds in higher dimensions.

3. Applications. As a corollary to Theorem 1, we have the following result concerning the case $n = 2$.

COROLLARY 4 (YOSHINAGA-SUZUKI [**18**], NISHIMURA [**7**]). *Let $f(x,y)$ (resp. $g(x,y)$) be a weighted homogeneous polynomial with weights (w_1,w_2) (resp. (w'_1,w'_2)) with an isolated singularity. If $(\mathbf{C}^2, f^{-1}(0))$ is locally homeomorphic to $(\mathbf{C}^2, g^{-1}(0))$ at the origin, then we have $w_i = w'_i$ $(i = 1,2)$ up to order.*

To prove Corollary 4, we need the following.

LEMMA 5. *Let f and g be polynomials in* $\mathbf{C}^2$ $(f(0) = g(0) = 0)$ *with isolated singularities at the origin. If* $(\mathbf{C}^2, f^{-1}(0))$ *is locally homeomorphic to* $(\mathbf{C}^2, g^{-1}(0))$ *at the origin, then* (S^3_ε, K_f) *is relatively homeomorphic to* (S^3_ε, K_g).

One can prove Lemma 5 using the following two facts and Lemma 2 (ii). The detailed proof is omitted.

FACT 1 [**1**]. If f and g are irreducible polynomials and $\Delta_f(t) = \Delta_g(t)$, then (S^3_ε, K_f) is relatively homeomorphic to (S^3_ε, K_g).

FACT 2 [**20**]. Let $f = p_1 \cdots p_r$ and $g = q_1 \cdots q_r$ be irreducible factorizations. If $(S^3_\varepsilon, K_{p_i})$ is relatively homeomorphic to $(S^3_\varepsilon, K_{q_i})$ $(1 \le i \le r)$ and the linking number of K_{p_i} and K_{p_j} in S^3_ε is the same as that of K_{q_i} and K_{q_j} $(1 \le i,\ j \le r,\ i \ne j)$, then (S^3_ε, K_f) is relatively homeomorphic to (S^3_ε, K_g).

PROOF OF COROLLARY 4. Set $f_1(x,y,z) = f(x,y) + z^2$ and $g_1(x,y,z) = g(x,y) + z^2$. Then f_1 (resp. g_1) is weighted homogeneous with weights $(w_1, w_2, 2)$ (resp. $(w'_1, w'_2, 2)$) with an isolated singularity. Note that (S^5, K_{f_1}) (resp. (S^5, K_{g_1})) is the 2-fold cyclic suspension of (S^3, K_f) (resp. (S^3, K_g)) [**5**]. Since (S^3, K_f) is relatively homeomorphic to (S^3, K_g) by Lemma 5, (S^5, K_{f_1}) is relatively homeomorphic to (S^5, K_{g_1}). Since $(\mathbf{C}^3, f_1^{-1}(0))$ (resp. $(\mathbf{C}^3, g_1^{-1}(0))$) is locally the cone over (S^5, K_{f_1}) (resp. (S^5, K_{g_1})) [**4**], $(\mathbf{C}^3, f_1^{-1}(0))$ is locally homeomorphic to $(\mathbf{C}^3, g_1^{-1}(0))$ at the origin. Thus by Theorem 1, f_1 and g_1 have the same weights. This implies $w_i = w'_i$ $(i = 1, 2)$ up to order.

Next we consider an application to Zariski's question [**19**]: Is the multiplicity of a hypersurface singularity a topological invariant?

LEMMA 6. *Let* $f(z_1, \ldots, z_n)$ *be a weighted homogeneous polynomial with weights* $(w_1, \ldots, w_n)$ *with an isolated singularity. Then the multiplicity of f at the origin is equal to* $\min\{m \in \mathbf{Z}; m \ge w\}$, *where* $w = \min\{w_1, \ldots, w_n\}$.

Lemma 6 follows from the definition of weights and Korollar 1.6 of [**16**]. The detailed proof is omitted.

Using Lemma 6 and Theorem 1, we have the following immediately.

PROPOSITION 7. *Let* $f(x,y,z)$ *(resp.* $g(x,y,z)$*) be a weighted homogeneous polynomial in* $\mathbf{C}^3$ *with an isolated singularity. If* $(\mathbf{C}^3, f^{-1}(0))$ *is locally homeomorphic to* $(\mathbf{C}^3, g^{-1}(0))$ *at the origin, then f and g have the same multiplicity at the origin.*

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