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A KÄHLER DEFORMATION OF CP^n

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Dedicated to Professor Shun-ichi Tachibana on his 60*th birthday*

ABSTRACT. We shall construct a one-parameter family of complete Kähler metrics on the n-dimensional complex projective space CP^n, including the Fubini-study metric. We shall get, as a corollary, many noncanonical SC^m Kähler structures on CP^n.

1. Introduction. There are many complete Kähler metrics on the n-dimensional complex Euclidean space C^n and on the n-dimensional complex hyperbolic space CH^n. But on the n-dimensional complex projective space CP^n ($n \geq 2$) there have been few results on such metrics except the Fubini-Study metric.

In this paper, we shall prove the following results.

THEOREM. *Let J be the canonical complex structure of the n-dimensional complex projective space CP^n and m a point of CP^n. Then, for a positive number ε, there exists a one-parameter family of complete Kähler metrics ds_a^2 $(-\varepsilon < a < \varepsilon)$ on CP^n compatible with the structure J satisfying the following properties.*

(1) *For $a = 0, ds_a^2 = ds_{can}^2$, where ds_{can}^2 is the Fubini-Study metric on CP^n.*

(2) *For each a ($|a| < \varepsilon$), $ds_a^2 = ds_{can}^2$ on $W_1 \cup W_2$, where W_1 (resp. W_2) is a neighborhood of the point m (resp. of the cut locus of the point m with respect to ds_{can}^2).*

(3) *For each a ($|a| < \varepsilon$), (CP^n, ds_a^2, J) is unitary-symmetric at the point m.*

(4) *For different values a, b in $(-\varepsilon, \varepsilon)$ and for each positive number λ, $ds_a^2 \neq \lambda\, ds_b^2$.*

COROLLARY. *There exist noncanonical SC^m Kähler structures on CP^n.*

We are in the C^∞ category, unless otherwise stated, and refer the readers to the following section for the terminology.

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2. Proof of the Theorem. In this section, we shall recall the notion of a unitary-symmetric Kähler manifold and of a SC^m manifold (see [**1, 4**] for detail), and then prove the Theorem.

Let $\gamma = \gamma(r)$ $(0 \leq r \leq l)$ be a geodesic in a connected, complete, n-dimensional Riemannian manifold (M, g). γ is said to be closed if $\gamma(0) = \gamma(l)$ and $\dot\gamma(0) = \dot\gamma(l)$, where $\dot\gamma(r)$ is the velocity vector of γ at r, and γ is said to be simple if it is one-to-one

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on the interval $[0, l)$. A Riemannian manifold (M, g) is said to be a SC^m manifold if all the geodesics starting from a point m in M are simple closed geodesics with constant length l.

A connected, Kähler manifold (M, g, J) of complex dimension n is said to be unitary-symmetric at a point m in M if the linear isotropy group at the point m of the group of all isometrically biholomorphic mappings of (M, g, J) onto itself is $U(n) \equiv U(T_m(M))$, where $T_m(M)$ is the tangent space to M at m.

We shall now prove the Theorem. Let c, $2/3 < c < 1$, be a constant and, for a real parameter a, varying in a neighborhood of zero, define the function $h_a(t)$ on the interval $(-1, \infty)$ by

$$h_a(t) := \log(t+1) + au(t),$$

where $u(t) := \exp[1/(1-c-t)+1/(t-c-1)]$ for $|t-1| < c$, and zero for $-1 < t \leq 1-c$ or $1+c \leq t$. Then it can be easily seen that $h_a(t)$ has the following properties: There exists a positive constant ε depending only on c such that for each a ($|a| < \varepsilon$),

$$\dot{h}_a(t) > 0, \qquad \dot{h}_a(t) + t\ddot{h}_a(t) > 0 \tag{2.1}$$

on $(-1, \infty)$, where $(\dot{\ })$ means differentiation with respect to t. Using this perturbed function $h_a(t)$ of the local potential function for the Fubini-Study metric on CP^n we shall construct a one-parameter family of Kähler metrics ds_a^2 on CP^n which satisfy the conditions of the Theorem. We denote by $(w^0 : w^1 : \cdots : w^n)$ the homogeneous coordinate system of CP^n and by U the open subset of CP^n defined by $w^0 \neq 0$. Then on U we take $z^\alpha = w^\alpha / w^0$, $\alpha = 1, \ldots, n$, as a complex local coordinate system in CP^n. We define a tensor field of type $(0, 2)$ on U by

$$ds_a^2 = 2\sum (g_a)_{\alpha\bar{\beta}}\, dz^\alpha\, d\bar{z}^\beta, \tag{2.2}$$

where $(g_a)_{\alpha\bar{\beta}} = \partial^2 h_a(\sum z^\gamma \bar{z}^\gamma)/\partial z^\alpha \partial \bar{z}^\beta$. Then from (2.1) we see that for each a ($|a| < \varepsilon$), ds_a^2 is a Kähler metric on the subset U and that outside the closed subset $K = \{(w^0 : w^1 : \cdots : w^n);\ w^0 \neq 0,\ \sum |w^\alpha / w^0|^2 \in [1-c, 1+c]\}$ of U, ds_a^2 is always equal to the Fubini-Study metric ds_{can}^2 on CP^n whose holomorphic sectional curvature is 4 (cf. [**2**]). From this observation, ds_a^2 can be extended to a Kähler metric on CP^n satisfying $ds_a^2 = ds_{can}^2$ on $CP^n - K$. It follows from (2.2) that the components $(g_a)_{\alpha\bar{\beta}}$ of ds_a^2 are written by

$$(g_a)_{\alpha\bar{\beta}} = \dot{h}_a(t)\delta_{\alpha\beta} + \ddot{h}_a(t)\bar{z}^\alpha z^\beta, \tag{2.3}$$

here and in what follows, we set $t = \sum z^\gamma \bar{z}^\gamma$. Putting $G_a = \det((g_a)_{\alpha\bar{\beta}})$, we have

$$G_a = (\dot{h}_a(t))^{n-1}[\dot{h}_a(t) + t\ddot{h}_a(t)].$$

Then, since the components $(R_a)_{\alpha\bar{\beta}}$ of the Ricci tensor of ds_a^2 are given by

$$(R_a)_{\alpha\bar{\beta}} = -\frac{\partial^2 \log G_a}{\partial z^\alpha \partial \bar{z}^\beta}$$

(cf. [**2**]), it follows that

$$(R_a)_{\alpha\bar{\beta}} = H_a(t)\delta_{\alpha\beta} + \dot{H}_a(t)\bar{z}^\alpha z^\beta \tag{2.4}$$

where $H_a(t) = -(d/dt)\log[(\dot{h}_a(t))^{n-1}\{\dot{h}_a(t) + t\ddot{h}_a(t)\}]$. Therefore, it follows from the property of the function $h_a(t)$ and (2.3) that the function $l(t,a)$ $(0 \le t < \infty)$ defined by

$$l(t,a) = H_a(t)/\dot{h}_a(t) - \dot{H}_a(t)/\ddot{h}_a(t) \tag{2.5}$$

is identically zero for $t \in [0, 1-c] \cup [1+c, \infty)$ and for a $(|a| < \varepsilon)$. But we see that for $t = 1$, $l(a) := l(1,a)$ is increasing in a $(|a| < \varepsilon)$, provided that ε is replaced by a smaller constant. In fact, putting $p = -\ddot{u}(1)/4 = c^{-3}\exp(-2/c)$, $q = u^{(4)}(1)/8 = 6c^{-6}(1-c)\exp(-2/c)$, we have

$$l(a) = 2a(1+16ap)^{-1}[(n-1)p(2+ap) + (1-ap)^{-2} \\ \times \{(13p - q) + ap(7p+q) - 2a^2p^3\}]$$

and $13p - q > 0$ when $2/3 < c < 1$.

Let us fix a small positive constant ε for which the condition (2.1) is satisfied for each a $(|a| < \varepsilon)$, and the function $l(a)$ is increasing in a $(|a| < \varepsilon)$. Suppose that $ds_a^2 = \lambda\, ds_b^2$ holds on CP^n, where a and b are in $(-\varepsilon, \varepsilon)$ and λ is a positive number. Then, from the fact that $ds_a^2 = ds_{can}^2$ on the subset $CP^n - K$, it follows immediately that $\lambda = 1$. Considering ds_a^2 on the subset K and using (2.3)–(2.5) and $\lambda = 1$, we have, in particular, $l(a) = l(b)$. It then follows from the choice of ε that $a = b$. Hence we see that ds_a^2 $(|a| < \varepsilon)$ is a family of complete Kähler metrics on CP^n, satisfying the assertions of the Theorem except for (3).

We now show the assertion (3) of the Theorem. First, note that each $\sigma \in U(n)$ can be identified with an element of $U(n+1)$ satisfying $\sigma(m) = m$, $m = (1 : 0 : \cdots : 0) \in CP^n$ and is an isometrically biholomorphic mapping of (CP^n, ds_a^2, J) onto itself for each a in $(-\varepsilon, \varepsilon)$. Since each $\sigma \in U(n)$ acts on CP^n linearly through the complex local coordinate system $\{z^\alpha\} = \{w^\alpha/w^0\}$, $(\sigma_*)_m$, the differential of the mapping σ at m, may be canonically identified with σ itself. This implies that for each a $(|a| < \varepsilon)$, (CP^n, ds_a^2, J) is unitary-symmetric at the point m. The proof of the Theorem is completed.

Now the Corollary follows from the assertion (3) of the Theorem and Theorem 2 in [**3**].

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