PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 103, Number 3, July 1988

A NONORIENTABLE COMPLETE MINIMAL SURFACE IN R^3 BETWEEN TWO PARALLEL PLANES

FRANCISCO J. LOPEZ

(Communicated by David G. Ebin)

ABSTRACT. In this paper, we show how to produce nonorientable, regular complete, minimal surfaces in R^3 between two parallel planes.

Introduction. Recently, great progress has been made in the classical theory of minimal surfaces in R^3. One of the problems that has been studied the last years is the kind of bounds that these surfaces admit in R^3.

So, in 1980, Jorge and Xavier [**1**] showed examples of orientable, regular, complete, minimal surfaces in R^3, with a coordinate bounded. The main tool used was Runge's Theorem, improving Weierstrass' representation of orientable minimal surfaces in R^3.

The aim of this work is to show that the orientability is not essential, and construct a family of regular, complete, minimal Möbius strips contained in a slab of R^3.

We use Runge's Theorem too, and Weierstrass' representation of nonorientable minimal surfaces in R^3, due to Meeks [**2**].

1. Preliminaries. Let $C(1/L, L)$ be the open annulus in $\mathbf{C}$ with outer and inner radii L and $1/L$, respectively, where $L > 1$. We write $C = C(1/L, L)$ for convenience.

We need the following lemmas.

LEMMA 1 [**3**]. *Let $f, g: C \to \mathbf{C}$ be two functions, f being holomorphic and g being meromorphic, such that when a pole of order m of g occur, f has a zero of order $2m$.*

We suppose that the holomorphic functions on C:

$$(1) \qquad \phi_1 = \frac{f}{2}(1-g^2), \quad \phi_2 = i\frac{f}{2}(1+g^2), \quad \phi_3 = fg$$

do not have real periods.

Then $x = (\operatorname{Re}\int(\phi_1), \operatorname{Re}\int(\phi_2), \operatorname{Re}\int(\phi_3))$ defines a regular, minimal immersion of C in R^3. Moreover, the element of length is given by $ds = \lambda|dz|$, where $\lambda = (|f|/2)(1+|g|^2)$.

Received by the editors June 1, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 53A10.

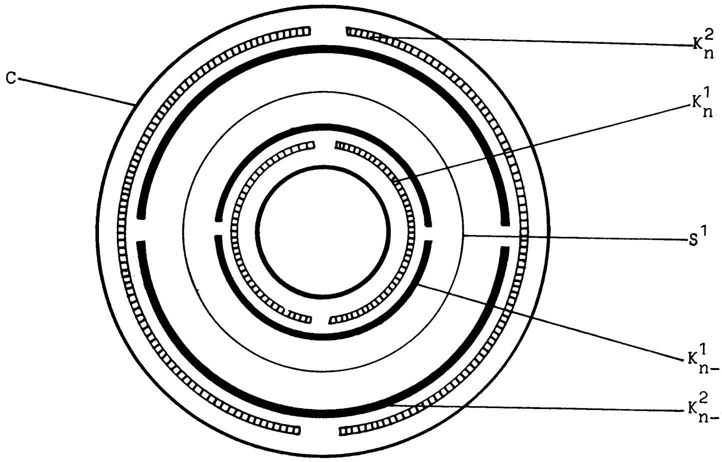

FIGURE 1

LEMMA 2 [**2**]. *Under the assumption of Lemma* 1, *the minimal surface is the double surface of a nonorientable, regular, minimal surface in* R^3 *if and only if*

$$(2)\qquad \begin{aligned}&\text{(i) } g(I(z)) = I(g(z)),\\ &\text{(ii) } (zg(z))^2 = -(\overline{f}(I(z)))/f(z),\end{aligned}$$

where $I\colon C \to C$ *is defined by* $I(z) = -1/\overline{z}$.

The nonorientable surface is, concretely, the Möbius strip $C/\{1, I\}$.

Now consider Figure 1.

As indicated, K_n^2 is the compact region formed by deleting two antipodal pieces centered at the imaginary axe, when n is even, and at the real axe when n is odd, to an annulus contained in C. Notice that $-K_n^2 = K_n^2$.

We observe that K_n^2 tend to the outer circle of C when $n \to \infty$.

At last, we define $K_n^1 = I(K_n^2)$, and $K_n = K_n^1 \cup K_n^2$.

For each $n \in N$, let $M_n, N_n \in \mathbf{C}$ and fix $\varepsilon > 0$.

LEMMA 3. *There exists a holomorphic function h on C such that*

$$(3)\qquad |h - M_n| \le \varepsilon \quad \text{on } K_n^2 \qquad \text{and} \qquad |h - N_n| \le \varepsilon \quad \text{on } K_n^1.$$

PROOF. Our main tool will be Runge's Theorem. We argue by an induction process.

By Runge's Theorem, there exists a holomorphic function on C h_1 such that

$$|h_1 - M_1| < \frac{\varepsilon}{2} \quad \text{on } K_1^2 \qquad \text{and} \qquad |h_1 - N_1| < \frac{\varepsilon}{2} \quad \text{on } K_1^1.$$

If we have constructed a holomorphic function on C h_{n-1} satisfying $|h_{n-1} - M_{n-1}| < \varepsilon/2^{n-1}$ on K_{n-1}^2, $|h_{n-1} - N_{n-1}| < \varepsilon/2^{n-1}$ on K_{n-1}^1, $|h_{n-1} - h_{n-2}| < \varepsilon/2^{n-1}$ on D_{n-2} where D_{n-2} is a closed annulus in C such that $D_{n-2} \cap (\bigcup_j K_j) = \bigcup_{i=1}^{n-2} K_i$, then we construct by Runge's Theorem a holomorphic function h_n on C such that $|h_n - M_n| < \varepsilon/2^n$ on K_n^2, $|h_n - N_n| < \varepsilon/2^n$ on K_n^1, $|h_n - h_{n-1}| < \varepsilon/2^n$ on D_{n-1} where D_{n-1} is a closed annulus in C satisfying: $D_{n-1} \cap (\bigcup_j K_j) = \bigcup_{i=1}^{n-1} K_i$.

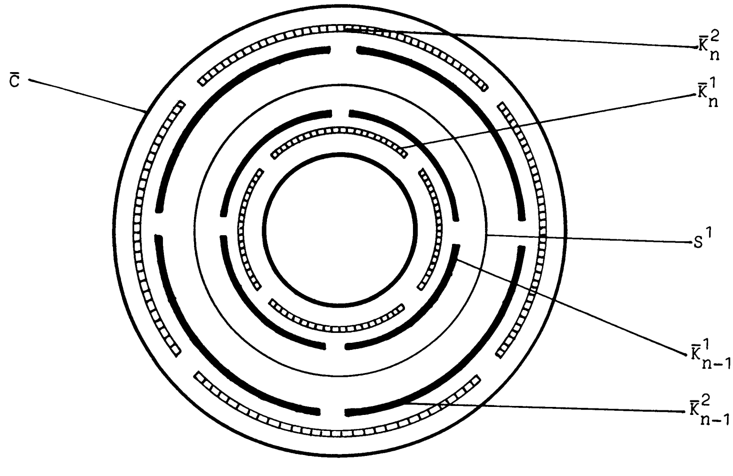

FIGURE 2

So, we obtain $\{h_n\}_{n\in N}$ a sequence of holomorphic functions on C.

This family is normal in Montel's sense, because $D_{n-1} \subset D_n$, $\bigcup_j D_j = C$ and $\max_{D_n} |h_j| \leq \varepsilon + \max\{\max_{D_n} |h_i|, i = 1, \ldots, n\}$, for every $j \in N$.

By Montel's Theorem, we can find a subsequence of $\{h_n\}_{n\in N}$ converging uniformly over compact subsets of C to a holomorphic function h on C.

By construction, h satisfies (3). Q.E.D.

2. Statement of result. We can now state our main theorem:

THEOREM. *There exists a family of regular, complete, minimal Möbius strips between two parallel planes in R^3.*

PROOF. Let $\{\alpha_n\}_{n\in N}$ be a sequence of real numbers, to be specified later.

We choose

$$M_{2n} = M_{2n-1} = \begin{cases} 1 & \text{if } n \text{ is even,} \\ \alpha_n & \text{if } n \text{ is odd,} \end{cases} \qquad N_{2n} = N_{2n-1} = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ \alpha_n & \text{if } n \text{ is even,} \end{cases}$$

and fix $\varepsilon > 0$.

By Lemma 3, there exists a holomorphic function h on C satisfying (3).

Define now $p(z) = e^{h(z)}$ and $t(z) = p(z)/\overline{p}(I(z))$ on C.

Consider

$$\overline{C} = C(1/L^{1/2}, L^{1/2}), \quad \text{and} \quad q(z) = t(z^2)t(-z^2) \quad \text{on } \overline{C}. \tag{4}$$

It is obvious that q is a holomorphic function on $\overline{C}$. Moreover, q satisfies

$$q(z)\overline{q}(I(z)) = 1. \tag{5}$$

We write $\overline{K}_n^i = \{z \in \overline{C}/z^2 \in K_n^i\}$, $n \in N$, $i = 1, 2$.

It is clear that $\{\overline{K}_n^2\}_{n\in N}$ is a sequence of compacts, which tend to the outer circle of $\overline{C}$ when $n \to \infty$, formed by deleting four pieces antipodes pairwise to an annulus, as in Figure 2 and $\overline{K}_n^1 = I(\overline{K}_n^2)$.

Using that $I(K_n^2) = K_n^1$, $-K_n^i = K_n^i$, $i = 1, 2$, (3) and the definition of q, we have

$$\text{(6)}\quad \begin{aligned} e^{2\alpha_n - 4\varepsilon - 2} \leq |q| \leq e^{2\alpha_n - 4\varepsilon + 2} \quad &\text{on } \overline{K}^2_{2n-1} \cup \overline{K}^2_{2n},\ n \text{ odd},\\ e^{2\alpha_n - 4\varepsilon - 2} \leq |q| \leq e^{2\alpha_n - 4\varepsilon + 2} \quad &\text{on } \overline{K}^1_{2n-1} \cup \overline{K}^1_{2n},\ n \text{ even}. \end{aligned}$$

Define now $f(z) = i[(z-1)^2/z^4]q(z)$, $g(z) = [z^2(z+1)/(z-1)](1/q(z))$ on $\overline{C}$.

By (4), Laurent's expansion series of q and $1/q$ on $\overline{C}$ are

$$\text{(7)}\qquad q(z) = \sum_{n\in Z} a_{4n} z^{4n} \quad \text{and} \quad \frac{1}{q(z)} = \sum_{n \in Z} a'_{4n} z^{4n}.$$

If we define ϕ_1, ϕ_2, ϕ_3 like in (1), (7) implies that ϕ_1, ϕ_2, ϕ_3 do not have real periods, and by Lemma 1, $x = (\operatorname{Re}\int\phi_1, \operatorname{Re}\int\phi_2, \operatorname{Re}\int\phi_3)$ gives a regular minimal immersion of $\overline{C}$ in R^3. We write this minimal surface by M.

We observe, by (5), that f and g satisfy (2), and Lemma 2 shows us that M is the double surface of a Möbius strip.

Moreover, notice that $|\phi_3| = |fg|$ is bounded in $\overline{C}$, and this implies that our surface is contained between two parallel planes, because $\operatorname{Re}\int\phi_3$ is bounded in $\overline{C}$.

We need only show that $\{\alpha_n\}_{n\in N}$ can be chosen so as to make M complete.

We know, by Lemma 1, that the element of length of M is given by

$$\text{(8)}\qquad ds = \lambda|dz|, \qquad \lambda = \frac{|f|}{2}(1+|g|^2) = \left[\frac{|z-1|^2}{2|z|^4}\right]|q(z)| + \frac{|z+1|^2}{2|q(z)|}.$$

The completeness will be proved verifying that the length in M of every divergent path is infinite, choosing a sequence of real numbers $\{\alpha_n\}_{n\in N}$ appropriate, like in [1].

Let $\alpha(t)$ be a divergent path in $\overline{C}$, where t is the Euclidean arclength of α.

We shall distinguish two cases.

(a) Suppose that α has infinite Euclidean length, i.e., $\alpha(t)\colon [0,\infty[\to \overline{C}$.

Since $\lambda \geq A(|q| + 1/|q|) \geq A$ on $M - K$, where K is a compact subset in $\overline{C}$ such that $\{1,-1\} \in K$, and A a positive constant, then

$$L(\alpha) = \int_0^\infty \lambda(t)\,dt = \infty.$$

(b) Suppose now that α has finite Euclidean length, i.e., $\alpha\colon [0,1[\to \overline{C}$.

Since α is divergent, we observe that α will cross all the $\overline{K}^2_{2n}$, or all the $\overline{K}^2_{2n-1}$, or all the $\overline{K}^1_{2n}$ or all the $\overline{K}^1_{2n-1}$, but a finite number.

Consider the first alternative, and let $m \in N$ such that α crosses every $\overline{K}^2_{2n}$, $n \geq m$. Let $J_n^i = \alpha \cap \overline{K}^i_n$, $n \in N$, $i = 1,2$.

Using (6) and (8), we have

$$L(\alpha) \geq \sum_{n\geq m} \int_{J^2_{2n}} \lambda(t)\,dt \geq \sum_{\substack{n \text{ odd} \\ n \geq m}} Le^{2\alpha_n} \text{ Length } (J^2_{2n})$$

with L a positive constant.

If r_n^i is the difference between the outer and inner radii of the annulus $\overline{K}_n^i$, $n \in N$, $i = 1, 2$, then $L(J_n^i) \geq r_n^i$, and therefore

$$L(\alpha) \geq \sum_{\substack{n \text{ odd} \\ n \geq m}} L e^{2\alpha_n} r_{2n}^2.$$

Choosing $\alpha_n \geq \max\{-Ln((r_{2n}^2)^{1/2}), -Ln((r_{2n-1}^2)^{1/2})\}$, n odd, we have that $L(\alpha) = \infty$ for every path α which crosses all the $\overline{K}_{2n}^2$ or all the $\overline{K}_{2n-1}^2$, but a finite number.

Similarly, choosing $\alpha_n \geq \max\{-Ln((r_{2n}^1)^{1/2}), -Ln((r_{2n-1}^1)^{1/2})\}$, n even, we have that $L(\alpha) = \infty$ if α crosses all the $\overline{K}_{2n}^1$ or all the $\overline{K}_{2n-1}^1$, but a finite number.

This choice of $\{\alpha_n\}_{n \in N}$ makes M complete. Q.E.D.

REFERENCES

1. L. P. de M. Jorge and F. Xavier, *A complete minimal surface in R^3 between two parallel planes*, Ann. of Math. (2) **112** (1980), 203–206.
2. W. H. Meeks III, *The classification of complete minimal surfaces in R^3 with total curvature greater than -8π*, Duke Math. J. **48** (1981), 523–535.
3. R. Osserman, *A survey of minimal surfaces*, Van Nostrand Reinhold, Princeton, N. J., 1969.

DEPARTAMENTO DE GEOMETRÍA Y TOPOLOGÍA, FACULTAD DE CIENCIAS, UNIVERSIDAD DE GRANADA, 18071-GRANADA, SPAIN