PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 103, Number 3, July 1988

CODIMENSION TWO NONORIENTABLE SUBMANIFOLDS WITH NONNEGATIVE CURVATURE

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(Communicated by David G. Ebin)

ABSTRACT. We prove that a compact nonorientable n-dimensional submanifold of $\mathbf{R}^{n+2}$ with nonnegative curvature is a "generalized Klein bottle" if $n \geq 3$.

1. Introduction. In [**1**] we study isometric immersions of compact, orientable nonnegatively curved n-manifolds in $\mathbf{R}^{n+2}$. The aim of this paper is to study the nonorientable case. If $n = 2$ there is an example of an isometric immersion of a flat Klein bottle in $\mathbf{R}^4$ (see [**3**]) and the results of [**1**] suggested that for $n \geq 3$ the "generalized Klein bottle" is, in fact, the only possible example. We will prove the following result.

THEOREM. *Let M^n, $n \geq 3$, be a compact, nonorientable Riemannian manifold with nonnegative sectional curvatures and $f\colon M^n \to \mathbf{R}^{n+2}$ an isometric immersion. Then there exists a $(n-1)$-dimensional manifold N^{n-1} homotopy equivalent to S^{n-1}, such that*

(1) *the orientable covering of M is diffeomorphic to $S^1 \times N^{n-1}$ and the metric is locally a product;*

(2) *M is diffeomorphic to a nonorientable bundle over S^1 with fibre N^{n-1} and the metric is locally a product;*

(3) *the covering projection sends the fibres N^{n-1} of $S^1 \times N^{n-1}$ isometrically onto the fibres of the bundle $M \to S^1$.*

2. Known facts. We will state now some results to be used in the proof of the theorem. For their proofs and related references see [**1**].

M will denote a n-dimensional Riemannian manifold, $n \geq 3$, compact, connected, with nonnegative sectional curvatures, which admits an isometric immersion $f\colon M^n \to \mathbf{R}^{n+2}$.

2.1. If M is orientable over a field F then $\sum_{i=1}^{n-1} b_i(M;F) \leq 2$, where $b_i(M;F) = \dim H_i(M;F)$ is the ith Betti number of M with coefficients in F.

2.2. If M is orientable, not simply connected, then $\pi_1(M)$ is cyclic, and if $n \geq 4$, $\pi_1(M) \cong Z$.

2.3. If M is orientable and $\pi_1(M) \cong Z$ then there exists a compact $(n-1)$-dimensional manifold N, homotopy-equivalent to a sphere such that M is diffeomorphic to $S^1 \times N$ and the metric is locally a product. In fact the universal covering of M is isometric to $\mathbf{R} \times N$.

Received by the editors July 13, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 53C40; Secondary 53C42.

3. Proof of the theorem.

3.1. $\pi_1(M) \cong Z$. Let $\theta\colon \overline{M} \to M$ be the orientation covering of M. By 2.2 $\pi_1(\overline{M})$ is cyclic. If $\pi_1(\overline{M})$ is finite then $b_1(M;\mathbf{R}) = 0$ and, by 2.2, $n = 3$. But M is not orientable and therefore $b_3(M;\mathbf{R}) = 0$ which leads to $0 = \chi(M) = 1 + b_2(M;\mathbf{R}) \geq 1$. So $\pi_1(\overline{M}) \cong Z$. Now, since θ is a double covering we have the exact sequence

$$0 \to Z \cong \pi_1(\overline{M}) \xrightarrow{\theta_\#} \pi_1(M) \to Z_2 \to 0.$$

It is not difficult to see that the only groups that fit such an exact sequence are Z, $Z \times Z_2$ and the semi-direct product $Z \ltimes_\phi Z_2$ where $\phi\colon Z_2 \to \mathrm{Aut}(Z) \cong Z_2$ is the identity. In the latter two cases we will have respectively

$$H_1(M;Z) \cong Z \oplus Z_2 \quad \text{and} \quad H_1(M;Z) \cong Z_2 \oplus Z_2$$

and in both cases $b_1(M;Z_2) = 2$. By duality $b_{n-1}(M;Z_2) = 2$ and therefore $\sum_{i=1}^{n-1} b_i(M;Z_2) \geq 4$ which contradicts 2.1. Therefore $\pi_1(M) \cong Z$ and $\theta_\#$ is multiplication by ± 2.

3.2. *M is a fibre bundle over S^1 with connected fibre.* By 2.3 $\overline{M}$ is diffeomorphic to $S^1 \times N$ and the metric is locally a product. Let $\overline{X}$ be a unitary vector field tangent to the S^1 factor. Then $\overline{X}$ is parallel and the only one up to a constant multiple (in fact, $H^1(\overline{M},\mathbf{R}) \cong \mathbf{R}$ is generated by a 1-form dual to a parallel field). Let $\tau\colon \overline{M} \to \overline{M}$ be the nontrivial covering transformation and define a vector field in M by

$$X(p) = \tfrac{1}{2}\{(d\theta)_x \overline{X}(x) + (d\theta)_{\tau(x)} \overline{X}(\tau(x))\}, \qquad p = \theta(x),$$

and then X is a well-defined parallel field. We want to show that $X \neq 0$. In fact if $X \equiv 0$ then $(d\theta)(\overline{X})$ defines a line field whose integral curves are projections of the integral curves of $\overline{X}$. More precisely consider $\theta^{-1}(p) = \{x, \tau(x)\}$ and let γ, σ be the integral curves of $\overline{X}$ through x and $\tau(x)$ respectively. If $\gamma \neq \sigma$, $\theta(\gamma)$ and $\theta(\sigma)$ represent the same closed curve in M with opposite orientation. Let α be a curve from x to $\tau(x)$. The loop $\gamma * \alpha * \sigma * \alpha^{-1}$ represent twice the generator of $\pi_1(\overline{M})$ and therefore is nonzero. But $\theta(\gamma * \alpha * \sigma * \alpha^{-1})$ is a commutator in $\pi_1(M)$, since, by the above, $\theta(\sigma) = \theta(\gamma)^{-1}$. This leads to a contradiction since $\pi_1(M)$ is abelian and $\theta_\#\colon \pi_1(\overline{M}) \to \pi_1(M)$ is $1-1$. If $\gamma = \sigma$ a similar argument leads to the same contradiction.

So X is a nonzero parallel field. The distribution $X^\perp$ is integrable and its leaves are the image, by θ, of the leaves of $\overline{X}^\perp$, so in particular they are compact. Now it is shown in [**2**], that for a complete Riemannian manifold M, there exist:

(a) A maximal subspace U of the space of parallel fields such that the leaves of $U^\perp$ are closed in M (Proposition III.5).

(b) A riemannian fibration of M on a m-dimensional flat torus, $m = \dim U$, whose fibres are the integral leaves of $U^\perp$ (Proposition III.6).

The space of parallel fields, in our case, is of dimension ≤ 1 (since $H_1(M;\mathbf{R}) \cong \mathbf{R}$), so it is spanned by X. So, with the above notation $U = \mathrm{span}\{X\}$ and the claim is proved.

3.3. *The fibres of $M \to S^1$ are homotopy spheres.* Let F be the fibre of the above fibration. Since F is connected and $\pi_1(M) \cong Z$ (by 3.1), we have the exact sequence

$$\pi_{j+1}(S^1) \to \pi_j(F) \to \pi_j(M) \to \pi_j(S^1) \to \cdots \to \pi_1(F) \to Z \to Z \to 0.$$

Therefore $\pi_1(F) = 0$ and $\pi_j(F) \cong \pi_j(M) \cong \pi_j(\overline{M}) \cong \pi_j(N)$, for $j \geq 2$ ($\overline{M} \cong S^1 \times N$), which prove 3.3 and the theorem.

4. Final remarks. The compact, nonorientable surfaces which admit metrics of nonnegative curvature are the flat Klein bottle and the projective plane $\mathbf{R}P^2$. As we mentioned in the introduction the flat Klein bottle admits an isometric immersion in $\mathbf{R}^4$. It would be interesting to know if $\mathbf{R}P^2$ admits an immersion in $\mathbf{R}^4$ with nonnegative curvature. Also it would be of interest to construct examples of "generalized Klein bottle" in $\mathbf{R}^{n+2}$ with nonnegative curvature.

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