CODIMENSION TWO NONORIENTABLE SUBMANIFOLDS WITH NONNEGATIVE CURVATURE
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ABSTRACT. We prove that a compact nonorientable n-dimensional submanifold of \( \mathbb{R}^{n+2} \) with nonnegative curvature is a "generalized Klein bottle" if \( n \geq 3 \).

1. Introduction. In [1] we study isometric immersions of compact, orientable nonnegatively curved n-manifolds in \( \mathbb{R}^{n+2} \). The aim of this paper is to study the nonorientable case. If \( n = 2 \) there is an example of an isometric immersion of a flat Klein bottle in \( \mathbb{R}^4 \) (see [3]) and the results of [1] suggested that for \( n \geq 3 \) the "generalized Klein bottle" is, in fact, the only possible example. We will prove the following result.

THEOREM. Let \( M^n, n \geq 3 \), be a compact, nonorientable Riemannian manifold with nonnegative sectional curvatures and \( f: M^n \rightarrow \mathbb{R}^{n+2} \) an isometric immersion. Then there exists a \((n - 1)\)-dimensional manifold \( N^{n-1} \) homotopy equivalent to \( S^{n-1} \), such that

1. the orientable covering of \( M \) is diffeomorphic to \( S^1 \times N^{n-1} \) and the metric is locally a product;
2. \( M \) is diffeomorphic to a nonorientable bundle over \( S^1 \) with fibre \( N^{n-1} \) and the metric is locally a product;
3. the covering projection sends the fibres \( N^{n-1} \) of \( S^1 \times N^{n-1} \) isometrically onto the fibres of the bundle \( M \rightarrow S^1 \).

2. Known facts. We will state now some results to be used in the proof of the theorem. For their proofs and related references see [1].

\( M \) will denote a n-dimensional Riemannian manifold, \( n \geq 3 \), compact, connected, with nonnegative sectional curvatures, which admits an isometric immersion \( f: M^n \rightarrow \mathbb{R}^{n+2} \).

2.1. If \( M \) is orientable over a field \( F \) then \( \sum_{i=1}^{n-1} b_i(M; F) \leq 2 \), where \( b_i(M; F) = \dim H_i(M; F) \) is the \( i \)th Betti number of \( M \) with coefficients in \( F \).

2.2. If \( M \) is orientable, not simply connected, then \( \pi_1(M) \) is cyclic, and if \( n \geq 4 \), \( \pi_1(M) \cong \mathbb{Z} \).

2.3. If \( M \) is orientable and \( \pi_1(M) \cong \mathbb{Z} \) then there exists a compact \((n - 1)\)-dimensional manifold \( N \), homotopy-equivalent to a sphere such that \( M \) is diffeomorphic to \( S^1 \times N \) and the metric is locally a product. In fact the universal covering of \( M \) is isometric to \( \mathbb{R} \times N \).

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3. Proof of the theorem.

3.1. \( \pi_1(M) \cong Z \). Let \( \theta : \overline{M} \to M \) be the orientation covering of \( M \). By 2.2 \( \pi_1(M) \) is cyclic. If \( \pi_1(M) \) is finite then \( b_1(M; \mathbb{R}) = 0 \) and, by 2.2, \( n = 3 \). But \( M \) is not orientable and therefore \( b_3(M; \mathbb{R}) = 0 \) which leads to \( 0 = \chi(M) = 1 + b_2(M; \mathbb{R}) \geq 1 \). So \( \pi_1(M) \cong Z \). Now, since \( \theta \) is a double covering we have the exact sequence

\[
0 \to Z \cong \pi_1(M) \xrightarrow{\theta^\#} \pi_1(M) \to Z_2 \to 0.
\]

It is not difficult to see that the only groups that fit such an exact sequence are \( Z \), \( Z \times Z_2 \) and the semi-direct product \( Z \ltimes_\phi Z_2 \) where \( \phi : Z_2 \to \text{Aut}(Z) \cong Z_2 \) is the identity. In the latter two cases we will have respectively

\[
H_1(M; Z) \cong Z \oplus Z_2 \quad \text{and} \quad H_1(M; Z) \cong Z_2 \oplus Z_2
\]

and in both cases \( b_1(M; Z_2) = 2 \). By duality \( b_{n-1}(M; Z_2) = 2 \) and therefore \( \sum_{i=0}^{n-1} b_i(M; Z_2) \geq 4 \) which contradicts 2.1. Therefore \( \pi_1(M) \cong Z \) and \( \theta^\# \) is multiplication by \( \pm 2 \).

3.2. \( M \) is a fibre bundle over \( S^1 \) with connected fibre. By 2.3 \( \overline{M} \) is diffeomorphic to \( S^1 \times N \) and the metric is locally a product. Let \( X \) be a unitary vector field tangent to the \( S^1 \) factor. Then \( X \) is parallel and the only one up to a constant multiple (in fact, \( H^1(M; \mathbb{R}) \cong \mathbb{R} \) is generated by a 1-form dual to a parallel field). Let \( r : M \to M \) be the nontrivial covering transformation and define a vector field in \( M \) by

\[
X(p) = \frac{1}{2} \left\{ (d\theta)_x X(x) + (d\theta)_{r(x)} X(r(x)) \right\}, \quad p = \theta^i(x),
\]

and then \( X \) is a well-defined parallel field. We want to show that \( X \neq 0 \). In fact if \( X \equiv 0 \) then \( (d\theta)(X) \) defines a line field whose integral curves are projections of the integral curves of \( X \). More precisely consider \( \theta^{-1}(x) = \{ x, \tau(x) \} \) and let \( \gamma, \sigma \) be the integral curves of \( X \) through \( x \) and \( \tau(x) \) respectively. If \( \gamma \neq \sigma \), then \( \theta(\gamma) \) and \( \theta(\sigma) \) represent the same closed curve in \( M \) with opposite orientation. Let \( \alpha \) be a curve from \( x \) to \( \tau(x) \). The loop \( \gamma \ast \alpha \ast \sigma \ast \alpha^{-1} \) represents twice the generator of \( \pi_1(M) \) and therefore is nonzero. But \( \theta(\gamma \ast \alpha \ast \sigma \ast \alpha^{-1}) \) is a commutator in \( \pi_1(M) \), since, by the above, \( \theta(\sigma) = \theta(\gamma)^{-1} \). This leads to a contradiction since \( \pi_1(M) \) is abelian and \( \theta^\#: \pi_1(M) \to \pi_1(M) \) is 1 - 1. If \( \gamma = \sigma \) a similar argument leads to the same contradiction.

So \( X \) is a nonzero parallel field. The distribution \( X^\perp \) is integrable and its leaves are the image, by \( \theta \), of the leaves of \( X^\perp \), so in particular they are compact. Now it is shown in [2], that for a complete Riemannian manifold \( M \), there exist:

(a) A maximal subspace \( U \) of the space of parallel fields such that the leaves of \( U^\perp \) are closed in \( M \) (Proposition III.5).

(b) A Riemannian fibration of \( M \) on a \( m \)-dimensional flat torus, \( m = \dim U \), whose fibres are the integral leaves of \( U^\perp \) (Proposition III.6).

The space of parallel fields, in our case, is of dimension \( \leq 1 \) (since \( H_1(M; \mathbb{R}) \cong \mathbb{R} \)), so it is spanned by \( X \). So, with the above notation \( U = \text{span}\{X\} \) and the claim is proved.

3.3. The fibres of \( M \to S^1 \) are homotopy spheres. Let \( F \) be the fibre of the above fibration. Since \( F \) is connected and \( \pi_1(M) \cong Z \) (by 3.1), we have the exact sequence

\[
\pi_{j+1}(S^1) \to \pi_j(F) \to \pi_j(M) \to \pi_j(S^1) \to \cdots \to \pi_1(F) \to Z \to Z \to 0.
\]
Therefore \( \pi_1(F) = 0 \) and \( \pi_j(F) \cong \pi_j(M) \cong \pi_j(\overline{M}) \cong \pi_j(N), \) for \( j \geq 2 \) (\( \overline{M} \cong S^1 \times N \)), which prove 3.3 and the theorem.

4. Final remarks. The compact, nonorientable surfaces which admit metrics of nonnegative curvature are the flat Klein bottle and the projective plane \( \mathbb{R}P^2 \).

As we mentioned in the introduction the flat Klein bottle admits an isometric immersion in \( \mathbb{R}^4 \). It would be interesting to know if \( \mathbb{R}P^2 \) admits an immersion in \( \mathbb{R}^4 \) with nonnegative curvature. Also it would be of interest to construct examples of “generalized Klein bottle” in \( \mathbb{R}^{n+2} \) with nonnegative curvature.

References