PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 103, Number 3, July 1988

A NOTE ON ${}_2sc(\mathbf{E}_1)$

RANA BARUA

(Communicated by Thomas J. Jech)

ABSTRACT. We exhibit a set of reals recursive in $\mathbf{E}_1$ which is not in $\mathscr{C}$, the smallest σ-field containing analytic sets and closed under operation $\mathscr{A}$. As a consequence, a conjecture of Hinman is shown to be false.

1. Introduction and preliminaries. The analogy between classical descriptive set theory on ω^ω and recursion theory was first firmly established by Addison (and independently by Mostowski) who perceived Kleene's hyperarithmetical relations as the effective analogue of the classical Borel sets. In [**5**], Hinman carried this analogy further by obtaining an effective analogue of the classical C-sets of Selivanovskii (the smallest σ-field containing closed sets and closed under the Suslin operation $\mathscr{A}$). Hinman obtained the effective hierarchy by generalizing Addison's construction of the "effective" Borel hierarchy which, roughly, consists of alternating applications of r.e. union and complementation. Briefly, Hinman's method consists of assigning to each set $C \subseteq \omega$ as it is generated an index or code $i(C)$ and at each stage of the inductive definition applying operation $\mathscr{A}$ to those sequences $\{F_n\}$ of subsets of ω for which $n \mapsto i(F_n)$ is recursive in some set previously generated. The class of sets thus obtained is exactly ${}_1sc(\mathbf{E}_1)$ (the class of subsets of ω recursive in $\mathbf{E}_1$), where $\mathbf{E}_1$ is the Tugué type-2 object associated with operation $\mathscr{A}$. This generalizes the result for the effective Borel sets (HYP), since HYP is precisely the class of sets recursive in ${}^2\mathbf{E}$, the Kleene's type-2 object associated with countable union.

Hinman's method does not seem to work for ω^ω. In his thesis [**4**], he obtained an effective hierarchy on ω^ω whose scope, however, forms only a proper subset of ${}_2sc(\mathbf{E}_1)$ (the class of subsets of ω^ω recursive in $\mathbf{E}_1$). One naturally asks whether it is possible to obtain a "reasonable" effective analogue of the classical C-sets on ω^ω which would exhaust ${}_2sc(\mathbf{E}_1)$. In this short note we show that this is not possible. This we prove by showing that a natural example of a set universal for the C-sets in ω^ω is actually recursive in $\mathbf{E}_1$. (Such a set was first constructed by Burgess and Lockhart in [**3**].) This implies that ${}_2sc(\mathbf{E}_1)$ is *not* a subset of the C-sets and consequently any reasonable effective version of the C-sets cannot exhaust ${}_2sc(\mathbf{E}_1)$. This incidentally shows that a conjecture of Hinman [**5**, p. 138] is *false.*

As in [**5**], we work with Hausdorff's $\delta - s$ operations, though it would be convenient to think of these operations as operation $\mathscr{A}$. (These can also be thought of as quantifiers.) A $\delta - s$ *operation* with base $N \subseteq \mathscr{P}(\omega)$ (power set of ω), written

Received by the editors April 22, 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 03D55, 03D65; Secondary 04A15.

Key words and phrases. Recursion in type-2 objects, effective hierarchy, C-sets, operation $\mathscr{A}$, $\delta - s$ operations, universal sets, games.

φ_N, is defined as

$$\varphi_N(\{A_n : n \in \omega\}) = \bigcup_{\eta \in N} \bigcap_{n \in \eta} A_n,$$

where $\{A_n : n \in \omega\}$ is a sequence of subsets of a set X.

Common examples of $\delta - s$ operations are countable union ($\cup$) and countable intersection ($\cap$) with bases $\{\eta \subseteq \omega : \eta \neq \varphi\}$ and $\{\omega\}$ respectively. The operation $\mathscr{A}$ is also a $\delta - s$ operation with a base consisting of all $\eta \subseteq \omega$ such that

$$\{\bar{\alpha}(n) = \langle \alpha(0), \ldots, \alpha(n-1) \rangle : n \in \omega\} \subseteq \eta, \quad \text{for some } \alpha \in \omega^\omega.$$

($\langle n_0, \ldots, n_{k-1} \rangle$ denotes the Gödel number or *sequence number* of $(n_0, \ldots, n_{k-1})$).

For any $\delta - s$ operation φ, $\nabla(\varphi)$ denotes the smallest family containing closed sets and closed under φ and complementation. Thus $\nabla(\cup) = \mathbf{\Delta}_1^1$ and $\nabla(\mathscr{A}) = C$-sets.

For any two $\delta - s$ operations φ and Ψ, we say that φ *subsumes* Ψ, in symbols $\varphi \geq \Psi$, if there exists a recursive function f such that for any family $\{A_n : n \in \omega\}$,

$$\Psi(\{A_n : n \in \omega\}) = \varphi(\{A_{f(n)} : n \in \omega\}).$$

Clearly operation $\mathscr{A}$ subsumes both $\cup$ and $\cap$.

With each $\delta - s$ operation φ (with base N) we associate a type-2 object $\mathbf{F}_\varphi$ (also writte $\mathbf{F}_N$) defined by

$$\mathbf{F}_\varphi(\alpha) = \begin{cases} 0 & \text{if } (\exists \eta \in N)(\forall n \in \eta)(\alpha(n) = 0), \\ 1 & \text{otherwise.} \end{cases}$$

2. Notation. We denote the set of natural numbers by ω. The letters $i, j, k, m, n, \ldots$ will stand for natural numbers. Seq will denote the set of sequence numbers of finite sequences of natural numbers. The letters $s, t, \ldots$ will denote finite sequences of natural numbers as well as their sequence numbers. The letter e will denotes the sequence number of the empty sequence. If $s, t \in \text{Seq}$, then $s * t$ denotes the sequence number of the concatenation of s followed by t; otherwise it is 0. If $s, t \in \text{Seq}$, then we write $s \subseteq t$ if t extends s (both considered as finite sequences).

The set of infinite sequences of natural numbers will be denoted by ω^ω. Elements of ω^ω will be called *reals* and are denoted by $\alpha, \beta, \ldots$. The letters ξ, η, with or without subscript, will denote subsets of ω. If $\alpha \in \omega^\omega$, then for each $n \in \omega$, $(\alpha)_n$ is a real such that $(\alpha)_n(m) = \alpha(\langle n, m \rangle)$ for all $m \in \omega$.

Unexplained notation and terminology from descriptive set theory are as in Moschovakis [**8**], while those from recursion theory are as in Hinman [**6**].

ACKNOWLEDGEMENT. I am grateful to V. V. Srivatsa for pointing out the import of the main result.

3. The main result. We now exhibit a set D which is universal for $\nabla(\varphi)$. We shall need the following relations.

$$\begin{aligned} S_1(\alpha) &\overset{\text{def}}{\leftrightarrow} \alpha \text{ codes some tree on } \omega \\ &\leftrightarrow (\forall s)[\alpha(s) = 0 \to (\text{Seq}(s) \& (\forall t)(\text{Seq}(t) \& t \subseteq s \to \alpha(t) = 0))]. \end{aligned}$$

$$S_2(\alpha) \overset{\text{def}}{\leftrightarrow} s \text{ is } \textit{terminal} \text{ for the tree coded by } \alpha$$
$$\leftrightarrow S_1(\alpha)\&\alpha(s) = 0\&(\forall n)(\alpha(s * \langle n\rangle) \neq 0).$$
$$\mathrm{WF}(\alpha) \overset{\text{def}}{\leftrightarrow} \alpha \text{ codes a wellfounded (wff) tree on } \omega$$
$$\leftrightarrow S_1(\alpha) \text{ and } (\forall\beta)(\exists i)\alpha(\bar\beta(i) \neq 0))$$
$$\leftrightarrow S_1(\alpha) \text{ and } (\forall\beta)(\exists i)S_2(\alpha, \bar\beta(i)).$$

Plainly, S_1 and S_2 are π^0_1 while WF is π^1_1.

If T is a wff tree on ω then $|T|$ denotes its length. If $T \neq \varnothing$, then $|T| = \rho(e)$, where ρ is the rank function (cf. [**8**, 2D]). If α codes a wff tree T then put $\|\alpha\| = |T|$.

Let X be a space of type 0 or 1 and let $G \subseteq \omega^\omega \times X$ be a good universal set for Σ^0_1 subsets of X. Define

$$H(\alpha, x) \leftrightarrow (G(\alpha^*, x)\&\alpha(0) = 0) \vee (\neg G(\alpha^*, x)\&\alpha(0) \neq 0),$$

where $\alpha^* = \lambda n \cdot \alpha(n+1)$.

Now, given a $\delta - s$ operation φ_N, we define $D(N; X) \subseteq \omega^\omega \times \omega^\omega \times X$ as follows.

$$\begin{aligned}(\sigma, \delta, x) \in D(N; X) \leftrightarrow{}& \mathrm{WF}(\sigma)\\ &\text{and } (\exists\eta_0 \in N)(\forall n_0 \in \eta_0)(\forall\eta_1 \in N)(\exists n_1 \in \eta_1)(\exists\eta_2 \in N)\\ &\cdots(\exists k)[S_2(\sigma, \underbrace{\langle n_0, \ldots, n_{k-1}\rangle}_{t})\&H((\delta)_t, x)].\end{aligned}$$

(As is usually the case, the infinite string of quantifiers is interpreted as a game played between two players $\forall$ and $\exists$. Cf. [**8**, 6D].)

Notice that the above equivalence can also be written as

$$\begin{aligned}(\sigma, \delta, x) \in D(N; X) \leftrightarrow{}& \mathrm{WF}(\sigma)\\ &\text{and } (\exists\eta_0 \in N)(\forall n_0 \in \eta_0)(\forall\eta_1 \in N)(\exists n_1 \in \eta_1)(\exists\eta_2 \in N)(\forall n_2 \in \eta_2)\\ &\cdots(\forall k)[S_2(\sigma, \underbrace{\langle n_0, \ldots, n_{k-1}\rangle}_{t}) \to H((\delta)_t, x)].\end{aligned}$$

This is because for any wff tree T and any sequence $\{m_i\}$ there is a unique i such that $\langle m_0, \ldots, m_{i-1}\rangle$ is terminal for T.

LEMMA 1. *$D(N; X)$ is universal for $\nabla(\varphi_N)$ subsets of X.*

PROOF. This follows from the arguments of §5(b) of [**2**].

LEMMA 2. *If φ_N subsumes $\mathscr{A}$, then $D(N, X)$ is recursive in $\mathbf{F}_N$.*

PROOF. Fix a recursive function $(\delta, s) \mapsto \delta^s$ from $\omega^\omega \times \omega \to \omega^\omega$ such that $(\delta^s)_t = (\delta)_{s*t}$. If β codes a tree T then let β_s be a code for $T_s = \{t\colon s * t \in T\}$ such that the function $(\beta, s) \mapsto \beta_s$ is recursive. Now define a partial function φ as follows.

$$\begin{aligned}\varphi(c, \sigma, \delta, x) &\simeq 0, \quad \text{if WF}(\sigma) \text{ and } \|\sigma\| = 0 \text{ and } H((\delta)_e, x);\\ &\simeq \mathbf{F}_N(\lambda k \cdot \mathbf{F}^0_N(\lambda k' \cdot \{c\}^{\mathbf{F}_N}(\sigma_{\langle k,k'\rangle}, \delta^{\langle k,k'\rangle}, x))),\\ &\qquad\qquad \text{if WF}(\sigma) \text{ and } \|\sigma\| > 0;\\ &\simeq 1, \quad \text{otherwise;}\end{aligned}$$

$$\text{where } \mathbf{F}^0_N(\alpha) = \begin{cases} 0, & \text{if } (\forall\eta \in N)(\exists n \in \eta)(\alpha(n) = 0);\\ 1, & \text{if } (\exists\eta \in N)(\forall n \in \eta)(\alpha(n) \neq 0).\end{cases}$$

Since φ_N subsumes $\mathscr{A}$, plainly WF is recursive in $\mathbf{F}_N$ and hence φ is recursive in $\mathbf{F}_N$. By the recursion theorem there is an index c^* such that

$$\varphi(c^*, \sigma, \delta, x) \simeq \{c^*\}^{\mathbf{F}_N}(\sigma, \delta, x).$$

We shall show by induction on $\|\sigma\|$ that $\{c^*\}^{\mathbf{F}_N}$ is the characteristic function of $D(N; X)$. The case $\|\sigma\| = 0$ is clear. Now suppose WF(σ) and $\|\sigma\| > 0$. Now

$$\begin{aligned}
&\{c^*\}^{\mathbf{F}_N}(\sigma, \delta, x) \simeq 0 \\
&\quad \text{iff } \mathbf{F}_N(\lambda k \cdot \mathbf{F}^0_N(\lambda k' \cdot \{c^*\}^{\mathbf{F}_N}(\sigma_{\langle k,k'\rangle}, \delta^{\langle k,k'\rangle}, x))) \simeq 0 \\
&\quad \text{iff } (\exists \eta' \in N)(\forall n' \in \eta')(\forall \eta'' \in N)(\exists n'' \in \eta'')[\{c^*\}^{\mathbf{F}_N}(\sigma_{\langle n',n''\rangle}, \delta^{\langle n',n''\rangle}, x) \simeq 0] \\
&\quad \text{iff } (\exists \eta' \in N)(\forall n' \in \eta')(\forall \eta'' \in N)(\exists n'' \in \eta'')[(\sigma_{\langle n',n''\rangle}, \delta^{\langle n',n''\rangle}, x) \in D(N; X)], \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{by induction hypothesis} \\
&\quad \text{iff } (\sigma, \delta, x) \in D(N; X), \quad \text{by definition of } D(N; X).
\end{aligned}$$

Then $\{c^*\}^{\mathbf{F}_N}$ is the characteristic function of $D(N; X)$ and so $D(N; X)$ is recursive in $\mathbf{F}_N$.

As an immediate consequence of the above we have

THEOREM. *Suppose Φ_N subsumes operation $\mathscr{A}$. Then ${}_2sc(\mathbf{F}_N)$ is not a subset of $\nabla(\mathbf{F}_N)$.*

PROOF. Since $\nabla(\mathbf{F}_N)$ is closed under complementation by the usual diagonal argument, the set

$$D' = \{\alpha \colon ((\alpha)_0, (\alpha)_1, \alpha) \in D(N; \omega^\omega)\}$$

is not in $\nabla(\mathbf{F}_N)$. But by Lemma 2, D' is recursive in $\mathbf{F}_N$.

REMARKS. For any $\delta - s$ operation Φ, it is quite reasonable to expect that any effective version of $\nabla(\Phi)$ would be a subset of $\nabla(\Phi)$. Thus if it were to exhaust ${}_2sc(\mathbf{F}_\Phi)$ then for $\Phi \geq \mathscr{A}$, this would have implied that ${}_2sc(\mathbf{F}_\Phi) \subseteq \nabla(\Phi)$, contradicting the above theorem.

This is precisely the reason why Hinman's conjecture [**5**, p. 138] is not true. In [**5**, §8], by adapting Moschovakis' definition of a hyperanalytic predicate [**7**], Hinman discussed a possible effective Γ-hierarchy over ω^ω for any $\delta - s$ operation Γ. For each $\alpha \in \omega^\omega$ one defines $I^\Gamma(\alpha)$, and for each $u \in I^\Gamma(\alpha)$ a set $[u; \Gamma, \alpha] \subseteq \omega$, just as in the case of the effective Γ-hierarchy over ω (briefly outlined in the introduction above) except that enumerations are taken relative to α as well as to some previously constructed set (cf. [**5**] for detail). Then for any ordinal λ recursive in $\mathbf{F}_\Gamma$, put

$$[u; \Gamma]^2_\lambda = \{\alpha \colon u \in I^\Gamma_{<\lambda}(\alpha) \text{ and } 0 \in [u; \Gamma, \alpha]\}.$$

These sets are easily seen to be recursive in $\mathbf{F}_\Gamma$ and Hinman conjectured that they exhaust ${}_2sc(\mathbf{F}_\Gamma)$. But it is not hard to see that each of the sets $[u; \Gamma]^2_\lambda$ is in $\nabla(\Gamma)$ and by our remarks made earlier, these sets cannot exhaust ${}_2sc(\mathbf{F}_\Gamma)$ at least for operations Γ subsuming $\mathscr{A}$.

In conclusion we would like to point out that one, however, obtains a hierarchy, which exhausts in ${}_1sc(\mathbf{F}_\Gamma)$ by taking sections of $D(\Gamma; \omega)$ at reals (β, δ) of increasing complexity. This is discussed in detail in [**1**].

REFERENCES

1. R. Barua, *Studies in set-theoretic hierarchies: from Borel sets to R-sets*, Ph.D. dissertation, Indian Statistical Institute, Calcutta, 1987.
2. J. P. Burgess, *Classical hierarchies from a modern standpoint*, Part I: *C-sets*, Fund. Math. **105** (1983), 81–95.
3. J. P. Burgess and R. Lockhart, *Classical hierarchies from a modern standpoint*, Part III: *BP-sets*, Fund. Math. **105** (1983), 107–118.
4. P. G. Hinman, *Ad astra per aspera: hierarchy schemata in recursive function theory*, Ph.D. dissertation, Univ. of California, Berkeley, 1966.
5. ——, *Hierarchies of effective descriptive set theory*, Trans. Amer. Math. Soc. **142** (1969), 111–140.
6. ——, *Recursion-theoretic hierarchies*, Springer-Verlag, Berlin and New York, 1978.
7. Y. N. Moschovakis, *Hyperanalytic predicates*, Trans. Amer. Math. Soc. **129** (1967), 249–282.
8. ——, *Descriptive set theory*, North-Holland, Amsterdam, 1980.

STATISTICS-MATHEMATICS DIVISION, INDIAN STATISTICAL INSTITUTE, 203 B.T. ROAD, CALCUTTA 700 035, INDIA