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PARTITIONER-REPRESENTABLE ALGEBRAS

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ABSTRACT. We give a simple proof of the theorem of Baumgartner and Weese on representability of Boolean algebras. We also show that the representability of $P(\omega_1)$ implies the existence of a relative Q_3-set.

In [**B-W**], Baumgartner and Weese introduced so called partition algebras: if E is a maximal antichain in $P(\omega)/\text{fin}$, then an element $a \in P(\omega)/\text{fin}$ is called a partitioner of E if, for each $e \in E$ either $e \leq a$ or $e \cdot a = 0$ and the subalgebra of all the partitioners factorized mod$[E]$ ($[E]$ denotes the ideal generated by the set E) is called the partition algebra of E. Finite sums $e_1 + \cdots + e_n$ of elements of E are called trivial partitioners. Of course, the notion of a partitioner of an antichain is meaningful in an arbitrary Boolean algebra.

Baumgartner and Weese proved the following

THEOREM 1. *Assuming* CH, *every Boolean algebra A, of cardinality $\leq c = 2^\omega$, is partitioner-representable i.e. it is isomorphic to the partition algebra of a maximal antichain E in $P(\omega)/\text{fin}$.*

We present below a simple proof of this theorem.

Observe that it suffices to construct a Parovičenko extension A^* of A and a antichain E in A^* such that the following two conditions are satisfied:

(1) Each nonzero element $a \in A$ is a nontrivial partitioner of E (nontrivial means that $e \leq a$ holds for infinitely many $e \in E$).

(2) Each element $b \in A^* \setminus A$ is either a nonpartitioner or is congruent mod$[E]$ to an element of A.

Indeed, because of Parovičenko characterisation (see e.g. [**C-N**]) A^* is (up to isomorphism) the algebra $P(\omega)/\text{fin}$, each partitioner $b \in A^*$ is congruent mod$[E]$ to a unique element $f(b)$ of A and the map f is then a homomorphism with kernel $[E]$. Note, that (1) and (2) imply that E is in fact a maximal antichain.

We define A^* and E as unions of increasing chains

$$A^* = \bigcup\{A_\alpha : \alpha < \omega_1\}, \quad E = \bigcup\{E_\alpha : \alpha < \omega_1\},$$

where the A_α's and E_α's are defined inductively, so that (the inductive assumption): each nonzero $a \in A$ is a partitioner of E_α but $a \notin [E_\alpha]$. We begin with $A_0 = A$, $E_0 = \varnothing$ and take unions at limit stages. It remains to describe the successor step. We may assume, that at each stage α we have fixed an enumeration of:

(a) All the nonzero elements of A_α.

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(b) All decreasing chains $b_0 > b_1 > \cdots$, in A_α.

(c) All chains of the form $b_0 > b_1 > \cdots > a_1 > a_0$, in A_α, so that at a given stage α we have an object x of type (a), (b) or (c) namely, the $K(\alpha)$th term of the $L(\alpha)$th enumeration (where K, L denote the inverses of a fixed pairing function: $\omega \times \omega_1 \to \omega_1$).

Now, given α, if x is as in (b) or (c) we put $E_{\alpha+1} = E_\alpha$ and extend A_α by adding a new element below the b_n's, in case (b) or in between the b_n's and a_n's, in case (c). To do this, identify A_α with the field $B(X_\alpha)$ of open-closed sets of the associated Stone space X_α and take $A_{\alpha+1}$ as the subfield of $P(X_\alpha)$ generated by $A_\alpha = B(X_\alpha)$ and the set $b = \bigcap\{b_n : n < \omega\}$.

If x is as in (a) we distinguish two cases. First, suppose $x \in A$. We may assume, that there is an atom $a \leq x$, disjoint from E_α, for otherwise we replace $A_\alpha = B(X_\alpha)$ by the field generated by A_α and $\{p\}$, where $p \in x \setminus \bigcup E_\alpha$ (note, that — by the inductive assumption — the family $\{x \setminus \bigcup s : s \in E_\alpha^{<\omega}\}$ is centered). Let $Y = (X_\alpha \setminus a) + Z$ (direct sum), where Z is one-point compactification of ω, and put $A_{\alpha+1} = B(Y)$.

Thus, $A_\alpha \subseteq A_{\alpha+1}$ since X_α is a continuous image of Y and in $A_{\alpha+1}$ there are infinitely many new atoms $e_0, e_1, \ldots$ below $a \leq x$ and all disjoint from E_α. Let $E_{\alpha+1} = E_\alpha \cup \{e_n : n < \omega\}$. Thus, each element of A remains a partitioner not in $[E_{\alpha+1}]$ and x becomes a nontrivial partitioner.

Finally, suppose $x \in A_\alpha \setminus A$ and x is a partitioner noncongruent mod$[E_\alpha]$ to elements of A (if $x \in A_\alpha \setminus A$ is not such, we add only a new atom under x and take $E_{\alpha+1} = E_\alpha$). Then, the complement $-x$ has the same property and the noncongruency condition implies that the ideal $I \subseteq A$, generated by the set $\{a \in A : a \cdot x \in [E_\alpha]$ or $a \cdot (-x) \in [E_\alpha]\}$ is proper. If p is an ultrafilter in A extending $-I = \{-a : a \in I\}$, then $\bigcap p \setminus \bigcup E_\alpha$ intersects both x and $-x$, in X_α. Hence, we can extend A_α by adding two new atoms: $e_0 \leq x$ and $e_1 \leq -x$, both disjoint from E_α and such that for each $a \in A$ either $e_0, e_1 \leq a$ or $e_0, e_1 \leq -a$. Defining $E_{\alpha+1}$ as $E_\alpha \cup \{e_0 + e_1\}$ we see that the inductive assumption holds for $E_{\alpha+1}$ and x becomes now a nonpartitioner. Q.E.D.

REMARK. As the referee pointed out, the Theorem above can be generalized for higher cardinalities as follows: Let κ be an infinite cardinal. For every Boolean algebra A, of cardinality $\leq 2^\kappa$ there is an extension A^* of cardinality 2^κ and a maximal antichain $E \subseteq A^*$ such that A^* satisfies the condition $H(\kappa^+)$ [**C-N**, p. 119] and A is isomorphic to the partition algebra of E.

We prove now, that — consistently — there is an algebra of cardinality c, which is not partitioner-representable. Recall that an uncountable subset X of Cantor space is said to be a Q_α-set, if each subset $Y \subseteq X$ is Borel, relatively in X, of order at most α. Miller, in [**M**], has constructed a model of ZFC in which there are no Q_α-sets, for all $\alpha < \omega_1$ and $2^\omega = 2^{\omega_1} = \omega_2$.

Hence, it suffices to prove the following

THEOREM 2. *If the algebra $P(\omega_1)$ is partitioner-representable, then there is a Q_3-set.*

PROOF. Let $A = P(\omega_1)$ (or, more generally, let A be complete, atomic). By assumption, there is maximal antichain $E \subseteq P(\omega)/\text{fin}$ and an isomorphism f from A onto the partition algebra of E. For each nonzero $a \in A$ we take a set $S(a) \subseteq \omega$

such that

$$f(a) = S(a)/\mathrm{fin}/[E].$$

Let us denote

$$E(a) = \{e \in E\colon e \leq S(a)/\mathrm{fin}\}.$$

Thus, for $a > 0$, $E(a)$ is infinite and

$$(*) \qquad \begin{gathered} a \leq b \text{ implies } E(a) \setminus E(b) \text{ is finite,} \\ a \circ b = 0 \text{ implies } E(a) \cap E(b) \text{ is finite.} \end{gathered}$$

Let $e_n^a = E_n^a/\mathrm{fin}$, for $a > 0$ and $n < \omega$, be distinct elements from $E(a)$. The Cantor space C will be represented as $C = (2^\omega)^\omega$. To each a we assign an element $x^a = \langle x_n^a\colon n < \omega\rangle$ of C defined as follows: x_n^a is $\chi_{E_n^a}$, the characteristic function of the set E_n^a. By $(*)$ the set

$$X = \{x^a\colon a \text{ is an atom of } A\}$$

has cardinality ω_1. To see that X is a Q_3-set let

$$K_n(a) = \{x \in C\colon x_n \underset{*}{\leq} \chi_{S(a)}\}$$

(where $x \underset{*}{\leq} y$, for $x, y \in 2^\omega$, means that the set $\{n\colon x(n) > y(n)\}$ is finite) and let

$$q(a) = \bigcap_{n<\omega} \bigcup_{i>n} K_i(a).$$

Thus, for each a, $q(a)$ is an $F_{\sigma\delta}$, in C and $x^a \in q(b)$ iff for infinitely many n we have $E_n^a \underset{*}{\subseteq} S(b)$ (i.e. $E_n^a \setminus S(b)$ is finite). From $(*)$ we obtain immediately

$$\begin{aligned} a \leq b &\quad \text{implies} \quad x^a \in q(b), \\ a \cdot b = 0 &\quad \text{implies} \quad x^a \notin q(b). \end{aligned}$$

Using the above formulas we infer at once, that for an arbitrary $Y \subseteq X$:

$$Y = q(b_Y) \cap X$$

where $b_Y = \bigcup\{a\colon x^a \in Y\}$. Q.E.D.

REFERENCES

[B-W] J. E. Baumgartner and M. Weese, *Partition algebras for almost-disjoint sets*, Trans. Amer. Math. Soc. **274** (1982), 619–630.

[C-N] W. Comfort and S. Negrepontis, *The theory of ultrafilters*, Springer-Verlag, Berlin and New York, 1974.

[M] A. Miller, *On the length of Borel hierarchies*, Ann. Math. Logic **16** (1979), 233–267.

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