TRANSIENCE OF A PAIR OF LOCAL MARTINGALES

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(Communicated by Daniel W. Stroock)

ABSTRACT. We consider the process of windings of complex Brownian motion about two points \(a\) and \(b\) in the complex plane, \(\{(\theta^a(t), \theta^b(t)) : t \geq 0\}\). We show that this process is transient in the sense that \(\lim_{t \to \infty} |(\theta^a(t), \theta^b(t))| = \infty\).

This extends a result found in both Lyons and McKean (1984) and McKean and Sullivan (1984). We will mostly use facts and ideas found in the former paper.

Introduction. In the following we will consider Brownian motion \(\{B(t) : t \geq 0\}\) starting at the origin \(O\). Planar Brownian motion a.s. does not hit points distinct from its starting place. Therefore we may define the continuous processes

\[
\{\theta^a(t) : t \geq 0\} = \{\arg(B(t) - a) : t \geq 0\}
\]

and

\[
\{\theta^b(t) : t \geq 0\} = \{\arg(B(t) - b) : t \geq 0\}
\]

for any two points \(a\) and \(b\) distinct from \(O\). If we further specify that the two processes will have initial values in the interval \([0, 2\pi)\) then a.s. the processes are uniquely defined.

We intend to show that for any two such points \(a\) and \(b\) which are distinct:

\[
\lim_{t \to \infty} |(\theta^a(t), \theta^b(t))| = \infty.
\]

We will prove this for the special case \(a = 1, b = -1\); that is we will prove

**PROPOSITION 1.** If we define the continuous process \(\{(\theta^1(t), \theta^{-1}(t)) : t \geq 0\}\) such that for each \(t\)

\[
(\theta^1(t), \theta^{-1}(t)) = (\arg(B(t) - 1), \arg(B(t) + 1))
\]

for a continuous planar Brownian motion, then

\[
\lim_{t \to \infty} |(\theta^1(t), \theta^{-1}(t))| = \infty.
\]

This particular result will establish the more general result as \(C \setminus \{a, b\}\) is conformally equivalent to \(C \setminus \{-1, 1\}\) for any distinct \(a\) and \(b\).

If we lift the Brownian motion \(\{B(t) : t \geq 0\}\) up to the class surface of \(C \setminus \{-1, 1\}\), then we obtain a Brownian motion \(\{Z(t) : t \geq 0\}\) such that

\[
Z(t) = Z(s) \quad \text{if and only if} \quad B(t) = B(s) \text{ and } (\theta^1(t), \theta^{-1}(t)) = (\theta^1(s), \theta^{-1}(s)).
\]
Lyons and McKeian (1984) and McKeian and Sullivan (1984) showed that this process was transient in the sense that $Z$ eventually quit every compact set of the class surface. This means that given any compact $K \subset C \setminus \{-1,1\}$ and any compact $N \subset \mathbb{R}^2$

$$\lim_{T \to \infty} P \left[ \text{there exists } t > T: B(t) \in K \& (\theta^1(t), \theta^{-1}(t)) \in N \right] = 0.$$  

This almost gives us our result but it leaves open the possibility that there exists a compact $N$ such that $(\theta^1(t), \theta^{-1}(t))$ visit $N$ during visits of $B(t)$ to small neighbourhoods of 1, −1 or infinity. We will show that this cannot happen.

1. The transience of Brownian motion on the class surface of $C \setminus \{-1,1\}$ enables us to reduce our problem as follows:

If $\{(\theta^1(t), \theta^{-1}(t): t \geq 0)\}$ is not transient then there exists some $(n(\theta), m(\theta))$ such that $\{t: (\omega^1(t), \omega^{-1}(t)) = (n(\omega)\pi, m(\omega)\pi)\}$ is unbounded.

It is easy to see that this latter statement is equivalent to

There exists $(n, m)$ such that with positive probability $\{t: (\theta^1(t), \theta^{-1}(t)) = (n\pi, m\pi)\}$ is unbounded.

It follows that our result will be proven if we can show for each $(n, m)$ that

$$P \left[ \text{there exists } T(\omega): \text{ for all } t > T(\omega) ((\theta^1(t), \theta^{-1}(t)) \neq (n\pi, m\pi)) \right] = 1.$$  

We will only prove this for the special case $(n, m) = (1, 0)$ but it will be clear that the proof works for all other pairs of integers.

We now recall some facts and definitions from Lyons and McKeian (1984).

A crossing of type 1 is an excursion of Brownian motion from the line segment $(-1,1)$ to $(1,\infty)$ without first hitting $(-\infty,-1)$ or an excursion from $(1,\infty)$ to $(-1,1)$ without first hitting $(-\infty,-1)$. We will denote the number of crossings of type 1 at time $t$ by $c_1(t)$.

A crossing of type 2 is an excursion of Brownian motion from the line segment $(-\infty,-1)$ to $(1,\infty)$ without first hitting $(-1,1)$ or an excursion from $(1,\infty)$ to $(-\infty,-1)$ without first hitting $(-1,1)$. We will denote the number of crossings of type 2 at time $t$ by $c_2(t)$.

A crossing of type 3 is an excursion of Brownian motion from the line segment $(-\infty,-1)$ to $(-1,1)$ without first hitting $(1,\infty)$ or an excursion from $(-1,1)$ to $(-\infty,-1)$ without first hitting $(1,\infty)$. We will denote the number of crossings of type 1 at time $t$ by $c_3(t)$.

The winding number of type 1 at time $t$, $\omega_1(t)$, is the number of crossings of type 1 by time $t$ which are anticlockwise minus the number which are clockwise.

The winding number of type 2 at time $t$, $\omega_2(t)$, is the number of crossings of type 2 by this time which are clockwise minus the number which are anticlockwise.

The winding number of type 3 at time $t$, $\omega_3(t)$, is the number of crossings of type 3 by this time which are anticlockwise minus the number which are clockwise.

Brownian motion in the plane is invariant if every excursion away from the real line is with probability $\frac{1}{2}$ reflected about the real line and with probability $\frac{1}{2}$ left unchanged; from this it can be seen that the numbers of windings of types 1, 2 and 3, given that the number of crossings of type $i = c_i$ for $i = 1, 2, 3$, are independently
distributed as \( c_i - 2 \text{Bin}(c_i, \frac{1}{2}) \), i.e. the difference between the number of heads and the number of tails from \( c_i \) tosses of a fair coin, where \( \text{Bin}(c_i, \frac{1}{2}) \) stands for the number of heads.

An important approximation is

\[
P\{c_i - 2 \text{Bin}(c_i, \frac{1}{2}) = n\} \leq \frac{K}{\sqrt{2\pi c_i}} \exp \left[ -\frac{n^2}{2c_i} \right]
\]

where \( K \) does not depend on \( c_i \) or \( n \). This inequality follows from Stirling's formula.

It is plain that for \( \theta^{-1}(t) \) to go from the value \((2n + 1)\pi\) to the value \((2n + 2)\pi\), the Brownian motion must cross from \((-\infty, -1)\) to either \((-1, 1)\) or \((1, \infty)\). That is, either \( \omega_1 \) must be incremented by 1 or \( \omega_2 \) must be decremented by 1. Similarly, for \( \theta^{-1}(t) \) to go from \((2n + 2)\pi\) either \( \omega_1 \) must be incremented by 1 or \( \omega_2 \) must be decremented by 1. The opposite holds for changes in the opposite direction.

Changes in \( \theta^1 \) are handled in a like manner. Thus, it is not difficult to see

(i) \( \theta^{-1}(t) \in (-\pi/2, \pi/2) \) only if \(|\omega_3(t) - \omega_2(t)|\) \(\leq 1\) and

(ii) \( \theta^1(t) \in (\pi/2, 3\pi/2) \) only if \(|\omega_1(t) - \omega_2(t)|\) \(\leq 1\).

Therefore if we can show that the state \(\{\omega_1(t) - \omega_2(t), \omega_3(t) - \omega_2(t)\} = (\alpha, \beta)\) is transient for each \((\alpha, \beta) \in \{1, 0, -1\}^2\), we will have proven the result. We shall in the following prove this for the case \((\alpha, \beta) = (0, 0)\). This is the same as proving that the state \(\{\omega_1(t) = \omega_2(t) = \omega_3(t)\}\) is transient.

**Proof that \(\{\omega_1(t) = \omega_2(t) = \omega_3(t)\}\) is transient.** Define \( c_a^1 \) and \( c_b^1 \) to be two concentric circles centred at 1 and of radius \( a \) and \( b \) respectively \((a < b)\). Let us likewise define the circles \( c_a^{-1} \) and \( c_b^{-1} \). Let the circles \( c_a^\infty \) and \( c_b^\infty \) to be the circles centred at 0 of radii \( 1/a \) and \( 1/b \) respectively. We choose \( a \) and \( b \) small enough so that none of the above circles intersects.

Define successively the stopping times

\[
T_0 = \inf\{t: B(t) \in c_a^1\},
S_i = \inf\{t > T_{i-1}: B(t) \in c_b^1\} \text{ for } i \geq 0,
T_i = \inf\{t > S_{i-1}: B(t) \in c_a^1\} \text{ for } i \geq 1.
\]

A time interval \([T_i, T_{i+1})\) will be called a loop about 1. The subinterval \([T_i, T_i)\) will be called the outward loop. The stochastic interval \([S_i, T_{i+1})\) will be called the inner loop.

We similarly define loops about \(-1\) and \(\infty\).

Given the transience of Brownian motion on the class surface of \(C\backslash\{1, -1\}\), it follows that we only have to show that \(\{\omega_1(t) = \omega_2(t) = \omega_3(t)\}\) cannot occur for infinitely many outward loops about 1, \(-1\) or \(\infty\). By the Borel-Cantelli Lemma, this will be accomplished by showing that

\[
\sum_{i=1}^{\infty} P_n^i < \infty \quad \text{for } i = 1, -1 \text{ or } \infty
\]

where

\[
P_n^i = \text{probability that } \omega_1 = \omega_2 = \omega_3 \text{ during an outward loop about } i.
\]

To show \((**)*\) we will use the following facts which are either taken from or derived from Lyons and McKean (1984).

**Fact 1.** There exist \(A, b, K > 0\) such that after \(n\) loops about 1, outside a set of probability \(< A e^{-b t}\).
The number of loops about $-1$ and $\infty$ are both in the interval $[n/K, Kn]$. Similarly with $1$ replaced by $-1$ or $\infty$.

**FACT 2.** After $n$ loops about $1 (-1, \infty), c_1$ the number of crossings of type 1 $(3,2)$ is greater than $n^2/\log^2 n$ with probability greater than $1 - De^{-g\log^2 n}$ for some $D, g > 0$.

Putting Facts 1 and 2 together we obtain

**FACT 3.** At the start of the $(n + 1)$th loop about $1 (-1, \infty)$, outside of a set of probability less than $3De^{-g\log^2 n} + Ae^{-bt}$ the minimum of $\{c_1, c_2, c_3\}$ is greater than $n^2/(K^2 \log^2 n)$.

For an outward loop about $1$ let $\maxwind$ be the maximum value obtained by:

$\omega(t) =$ number of anticlockwise crossings concluded during $[T_i, t)$-number of clockwise crossings concluded during $[T_i, t)$, for $t \in [T_i, T_{i+1})$ where $T_{i}, T_{i+1}$ are respectively the beginning and end of the outward loop in question. One may similarly define $\minwind$ for an outward loop.

Before beginning the final assault on Proposition 1 we require a lemma.

**LEMMA 1.** There exists $M$ such that $P \{ \text{During an outward loop maxwind is } \geq n \} \leq M/(n + 1)$ for all $n$.

**PROOF.** Without loss of generality suppose that the outward loop $([T_i, S_i])$ is about $1$. Consider $\{\theta^1(t): t \geq 0\}$, the continuous argument of $\{B(t): t \geq 0\}$. Maxwind will be greater than or equal to $n$ only if

$$\sup_{t \in (T_i, S_i)} \theta^1(t) - \theta^1(T_i) \geq (n + 1)\pi.$$ 

By the reflection principle

$$P \left[ \sup_{t \in (T_i, S_i)} \theta^1(t) - \theta^1(T_i) \geq (n + 1)\pi \right] = \frac{1}{2} P \left[ \theta^1(S_i) - \theta^1(T_i) \geq (n + 1)\pi \right].$$

But $\theta^1(S_i) - \theta^1(T_i)$ is Cauchy and so the result follows. Q.E.D.

The same result holds for minwind.

We are now ready to prove that $\sum_{i=1}^{\infty} P_n < \infty$. The proof for the other cases is exactly the same and so will not be given. The idea essentially is that during the outward loop $[T_i, S_i]$ the quantities $\omega_2(t)$ and $\omega_3(t)$ remain constant. So the only way that $\{\omega_1 = \omega_2 = \omega_3\}$ can occur within the outward loop is if for some $i$ and $j$

(i) $\omega_2(t) = \omega_3(t) = i, \omega_1(t) = j$

(ii) either maxwind or minwind is greater than $|i - j|$.

By approximation (#) of Lyons and McKean,

$$P\{\omega_2(t) = \omega_3(t) = i \text{ at the start of the } (n + 1)\text{th loop } |c_1(t), c_2(t), c_3(t)| < \frac{K}{\sqrt{c_2c_3}}e^{-i^2/2c_2}e^{-i^2/2c_3}.$$ 

Given that $\omega_2(t) = \omega_3(t) = i$ and $\omega_1(t) = j$ at the beginning of the $(n + 1)$th loop, we obtain from Lemma 1.1 that

$$P\{\omega_1 = \omega_2 = \omega_3 \text{ during the outward loop}\} \leq \frac{K}{|i - j| + 1}.$$
Therefore, given that $\omega_2 = \omega_3 = i$ and $(c_1(t), c_2(t), c_3(t))$ at the start of the outward loop, the probability that all three $\omega_k$ are equal during the outward loop is at most

$$K_1 \sum_{-\infty}^{\infty} \frac{1}{\sqrt{c_1}} e^{-j^2/2c_1} \frac{1}{|i - j| + 1}.$$ 

Using approximations similar to those in §1, we find that this is less than

$$K_2 \left\{ \frac{\log(|i| + 1)}{\sqrt{c_1}} + \frac{1}{|i| + 1} \right\}.$$ 

So the probability that $\{\omega_1 = \omega_2 = \omega_3\}$ during the $(n + 1)$th outward loop, given that the numbers of crossings at the start of the loop are $c_1, c_2, c_3$ respectively, is at most

$$K_3 \sum_{-\infty}^{\infty} \frac{1}{\sqrt{c_2 c_3}} e^{-i^2/2c_2} e^{-i^2/2c_3} \left[ \frac{\log(|i| + 1)}{\sqrt{c_1}} + \frac{1}{|i| + 1} \right]$$

$$\leq K_4 \left[ \frac{\log(c_3 c_2/(c_2 + c_3))}{\sqrt{c_2 + c_3}} \cdot \frac{1}{\sqrt{c_1}} + \frac{1}{\sqrt{c_2 c_3}} \right].$$

If $\min\{c_1, c_2, c_3\} \geq n^2/\log(n)^2$, then the above expression cannot be greater than $K_5((\log(n)^3)/n^2)$. We conclude that

$$P_n^1 \leq P \left\{ \min\{c_1, c_2, c_3\} < \frac{n^2}{\log(n)^2} \right\} + K_5 \frac{\log(n)^3}{n^2}$$

so

$$P_n^1 \leq 3K_1 e^{-h(\log(n)^2)} + 3K_2 e^{-hn} + K_3(\log(n)^3)/n^2.$$ 

Therefore $\sum P_n^1 < \infty$ and the proof is completed. Q.E.D.

2. The result which has just been proven has a geometric consequence.

The covering surface of $C\setminus\{1, -1\}$ can be thought of as the upper half plane, $H$. Each preimage of $C\setminus\{1, -1\} \cap \{z: \text{im}(z) > 0\}$ is in a 1-1 correspondence with an element of the homotopy group of $C\setminus\{1, -1\}$. Transience of Brownian motion on $H$ shows that the Brownian path in $C\setminus\{1, -1\}$ becomes more and more tangled up about the two deleted points as $t \to \infty$. See McKean (1969) for details, see also Durrett (1983) for a more direct proof of this fact.

Consider now a compact disc $D = \{z: |z - i| < 1/2\}$ in $C\setminus\{-1, 1\}$. The preimages of $D$ are also in a 1-1 correspondence with the elements of the homotopy group of $C\setminus\{1, -1\}$. Similarly there is a (not 1-1) correspondence between every preimage of $D$ and a branch of $(\arg(z - 1), \arg(z + 1))$ on $D$ such that for any branch $B$, the preimages of $D$ corresponding to $B$ are "dense" on the real line in the sense that

for each $x \in R^1$ and each $\epsilon > 0$ there exists a preimage of $D$ corresponding to $B$ in $\{z: |z - x| < \epsilon\}$.

The results of Lyons and McKean (1984) and McKean and Sullivan (1984) are equivalent to the statement that given a branch $B$, Brownian motion in $H$ eventually quits all the preimages of $D$ corresponding to $B$. The result of proposition tells us in addition that Brownian motion in $H$ a.s. cannot encircle preimages corresponding to $B$. By a result of Burgess Davis (1979), this means that for Lebesgue a.e. $x$ on $R^1$ and for each $\alpha \in (0, \pi/2)$, there exists $\epsilon(x)$ such that $\{\text{preimages of } D \text{ corresponding to } B\} \cap \{z: |z - x| < \epsilon; |\arg(z - x) - \pi/2| < \alpha\}$ is empty.
REFERENCES

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