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TRANSIENCE OF A PAIR OF LOCAL MARTINGALES

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ABSTRACT. We consider the process of windings of complex Brownian motion about two points a and b in the complex plane, $\{(\theta^a(t), \theta^b(t)) : t \geq 0\}$. We show that this process is transient in the sense that $\lim_{t\to\infty} |(\theta^a(t), \theta^b(t))| = \infty$. This extends a result found in both Lyons and McKean (**1984**) and McKean and Sullivan (**1984**). We will mostly use facts and ideas found in the former paper.

Introduction. In the following we will consider Brownian motion $\{B(t) : t \geq 0\}$ starting at the origin O. Planar Brownian motion a.s. does not hit points distinct from its starting place. Therefore we may define the continuous processes

$$\{\theta^a(t) : t \geq 0\} = \{\arg(B(t) - a) : t \geq 0\}$$

and

$$\{\theta^b(t) : t \geq 0\} = \{\arg(B(t) - b) : t \geq 0\}$$

for any two points a and b distinct from O. If we further specify that the two processes will have initial values in the interval $[0, 2\pi)$ then a.s. the processes are uniquely defined.

We intend to show that for any two such points a and b which are distinct:

$$\lim_{t\to\infty} |(\theta^a(t), \theta^b(t))| = \infty.$$

We will prove this for the special case $a = 1$, $b = -1$; that is we will prove

PROPOSITION 1. *If we define the continuous process* $\{(\theta^1(t), \theta^{-1}(t)); t \geq 0\}$ *such that for each* t

$$(\theta^1(t), \theta^{-1}(t)) = (\arg(B(t) - 1), \arg(B(t) + 1))$$

for a continuous planar Brownian motion, then

$$\lim_{t\to\infty} |(\theta^1(t), \theta^{-1}(t))| = \infty.$$

This particular result will establish the more general result as $C\backslash\{a, b\}$ is conformally equivalent to $C\backslash\{-1, 1\}$ for any distinct a and b.

If we lift the Brownian motion $\{B(t) : t \geq 0\}$ up to the class surface of $C\backslash\{-1, 1\}$, then we obtain a Brownian motion $\{Z(t) : t \geq 0\}$ such that

$$Z(t) = Z(s) \quad \text{if and only if} \quad B(t) = B(s) \text{ and } (\theta^1(t), \theta^{-1}(t)) = (\theta^1(s), \theta^{-1}(s)).$$

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Lyons and McKean (**1984**) and McKean and Sullivan (**1984**) showed that this process was transient in the sense that Z eventually quit every compact set of the class surface. This means that given any compact $K \subset C\backslash\{-1,1\}$ and any compact $N \subset R^2$

$$\lim_{T\to\infty} P\left[\text{there exists } t > T\colon B(t) \in K \& (\theta^1(t), \theta^{-1}(t)) \in N\right] = 0.$$

This almost gives us our result but it leaves open the possibility that there exists a compact N such that $(\theta^1(t), \theta^{-1}(t))$ visit N during visits of $B(t)$ to small neighbourhoods of $1, -1$ or infinity. We will show that this cannot happen.

1. The transience of Brownian motion on the class surface of $C\backslash\{-1,1\}$ enables us to reduce our problem as follows:

> If $\{(\theta^1(t), \theta^{-1}(t)\colon t \geq 0)\}$ is not transient then there exists some $(n(\theta), m(\theta))$ such that $\{t\colon (\omega^1(t), \omega^{-1}(t)) = (n(\omega)\pi, m(\omega)\pi)\}$ is unbounded.

It is easy to see that this latter statement is equivalent to

> There exists (n, m) such that with positive probability $\{t\colon (\theta^1(t), \theta^{-1}(t)) = (n\pi, m\pi)\}$ is unbounded.

It follows that our result will be proven if we can show for each (n, m) that

$$P\left[\text{there exists } T(\omega)\colon \text{ for all } t > T(\omega)(\theta^1(t), \theta^{-1}(t)) \neq (n\pi, m\pi)\right] = 1.$$

We will only prove this for the special case $(n, m) = (1, 0)$ but it will be clear that the proof works for all other pairs of integers.

We now recall some facts and definitions from Lyons and McKean (**1984**).

A *crossing of type* 1 is an excursion of Brownian motion from the line segment $(-1, 1)$ to $(1, \infty)$ without first hitting $(-\infty, -1)$ or an excursion from $(1, \infty)$ to $(-1, 1)$ without first hitting $(-\infty, -1)$. We will denote the number of crossings of type 1 at time t by $c_1(t)$.

A *crossing of type* 2 is an excursion of Brownian motion from the line segment $(-\infty, -1)$ to $(1, \infty)$ without first hitting $(-1, 1)$ or an excursion from $(1, \infty)$ to $(-\infty, -1)$ without first hitting $(-1, 1)$. We will denote the number of crossings of type 2 at time t by $c_2(t)$.

A *crossing of type* 3 is an excursion of Brownian motion from the line segment $(-\infty, -1)$ to $(-1, 1)$ without first hitting $(1, \infty)$ or an excursion from $(-1, 1)$ to $(-\infty, -1)$ without first hitting $(1, \infty)$. We will denote the number of crossings of type 1 at time t by $c_3(t)$.

The *winding number of type* 1 at time t, $\omega_1(t)$, is the number of crossings of type 1 by time t which are anticlockwise minus the number which are clockwise.

The *winding number of type* 2 at time t, $\omega_2(t)$, is the number of crossings of type 2 by this time which are clockwise minus the number which are anticlockwise.

The *winding number of type* 3 at time t, $\omega_3(t)$, is the number of crossings of type 3 by this time which are anticlockwise minus the number which are clockwise.

Brownian motion in the plane is invariant if every excursion away from the real line is with probability $\frac{1}{2}$ reflected about the real line and with probability $\frac{1}{2}$ left unchanged; from this it can be seen that the numbers of windings of types 1, 2 and 3, given that the number of crossings of type $i = c_i$ for $i = 1, 2, 3$, are independently distributed as $c_i - 2\,\mathrm{Bin}(c_i, \frac{1}{2})$, i.e. the difference between the number of heads and the number of tails from c_i tosses of a fair coin, where $\mathrm{Bin}(c_i, \frac{1}{2})$ stands for the number of heads.

An important approximation is

$$(\#)\qquad P\{c_i - 2\,\mathrm{Bin}(c_i, \tfrac{1}{2}) = n\} \leq \frac{K}{\sqrt{2\pi c_i}} \exp\left[-\frac{n^2}{2c_i}\right]$$

where K does not depend on c_i or n. This inequality follows from Stirling's formula.

It is plain that for $\theta^{-1}(t)$ to go from the value $(2n+1)\pi$ to the value $(2n+2)\pi$, the Brownian motion must cross from $(-\infty, -1)$ to either $(-1, 1)$ or $(1, \infty)$. That is, either ω_1 must be incremented by 1 or ω_2 must be decremented by 1. Similarly, for $\theta^{-1}(t)$ to go from $(2n+2)\pi$ either ω_1 must be incremented by 1 or ω_2 must be decremented by 1. The opposite holds for changes in the opposite direction. Changes in θ^1 are handled in a like manner. Thus, it is not difficult to see

(i) $\theta^{-1}(t) \in (-\pi/2, \pi/2)$ only if $|\omega_3(t) - \omega_2(t)| \leq 1$ and

(ii) $\theta^1(t) \in (\pi/2, 3\pi/2)$ only if $|\omega_1(t) - \omega_2(t)| \leq 1$.

Therefore if we can show that the state $\{(\omega_1(t) - \omega_2(t), \omega_3(t) - \omega_2(t)) = (\alpha, \beta)\}$ is transient for each $(\alpha, \beta) \in \{1, 0, -1\}^2$, we will have proven the result. We shall in the following prove this for the case $(\alpha, \beta) = (0, 0)$. This is the same as proving that the state $\{\omega_1(t) = \omega_2(t) = \omega_3(t)\}$ is transient.

PROOF THAT $\{\omega_1(t) = \omega_2(t) = \omega_3(t)\}$ IS TRANSIENT. Define c_a^1 and c_b^1 to be two concentric circles centred at 1 and of radius a and b respectively ($a < b$). Let us likewise define the circles c_a^{-1} and c_b^{-1}. Let the circles c_a^{∞} and c_b^{∞} to be the circles centred at 0 of radii $1/a$ and $1/b$ respectively. We choose a and b small enough so that none of the above circles intersects.

Define successively the stopping times

$T_0 = \inf\{t\colon B(t) \in c_a^1\}$,

$S_i = \inf\{t > T_i\colon B(t) \in c_b^1\}$ for $i \geq 0$,

$T_i = \inf\{t > S_{i-1}\colon B(t) \in c_a^1\}$ for $i \geq 1$.

A time interval $[T_i, T_{i+1})$ will be called a *loop about* 1. The subinterval $[T_i, S_i)$ will be called the *outward loop.* The stochastic interval $[S_i, T_{i+1})$ will be called the *inner loop.*

We similarly define loops about -1 and ∞.

Given the transience of Brownian motion on the class surface of $C\backslash\{1, -1\}$, it follows that we only have to show that $\{\omega_1(t) = \omega_2(t) = \omega_3(t)\}$ cannot occur for infinitely many outward loops about $1, -1$ or ∞. By the Borel-Cantelli Lemma, this will be accomplished by showing that

$$\sum_{i=1}^{\infty} P_n^i < \infty \quad \text{for } i = 1, -1 \text{ or } \infty$$

where

$$P_n^i = \text{probability that } \omega_1 = \omega_2 = \omega_3 \text{ during an outward loop about } i.$$

To show $(**)$ we will use the following facts which are either taken from or derived from Lyons and McKean (**1984**).

FACT 1. There exist $A, b, K > 0$ such that after n loops about 1, outside a set of probability $< Ae^{-bt}$.

The number of loops about -1 and ∞ are both in the interval $[n/K, Kn]$. Similarly with 1 replaced by -1 or ∞.

FACT 2. After n loops about 1 $(-1,\infty), c_1$ the number of crossings of type 1 $(3,2)$ is greater than $n^2/\log^2 n$ with probability greater than $1 - De^{-g\log^2 n}$ for some $D, g > 0$.

Putting Facts 1 and 2 together we obtain

FACT 3. At the start of the $(n+1)$th loop about 1 $(-1,\infty)$, outside of a set of probability less than $3De^{-g\log^2 n} + Ae^{-bt}$ the minimum of $\{c_1, c_2, c_3\}$ is greater than $n^2/(K^2\log^2 n)$.

For an outward loop about 1 let *maxwind* be the maximum value obtained by: $\omega(t) =$ number of anticlockwise crossings concluded during $[T_i, t)$-number of clockwise crossings concluded during $[T_i, t)$, for $t \in [T_i, T_{i+1})$ where T_i, T_{i+1} are respectively the beginning and end of the outward loop in question. One may similarly define *minwind* for an outward loop.

Before beginning the final assault on Proposition 1 we require a lemma.

LEMMA 1. *There exists M such that P {During an outward loop maxwind is $\geq n\} \leq M/(n+1)$ for all n.*

PROOF. Without loss of generality suppose that the outward loop $(= [T_i, S_i))$ is about 1. Consider $\{\theta^1(t) \colon t \geq 0\}$, the continuous argument of $\{B(t) \colon t \geq 0\}$. Maxwind will be greater than or equal to n only if

$$\sup_{t\in(T_i,S_i)} \theta^1(t) - \theta^1(T_i) \geq (n+1)\pi.$$

By the reflection principle

$$P\left[\sup_{t\in(T_i,S_i)} \theta^1(t) - \theta^1(T_i) \geq (n+1)\pi\right] = \tfrac{1}{2}P\left[\theta^1(S_i) - \theta^1(T_i) \geq (n+1)\pi\right].$$

But $\theta^1(S_i) - \theta^1(T_i)$ is Cauchy and so the result follows. Q.E.D.

The same result holds for minwind.

We are now ready to prove that $\sum_{i=1}^{\infty} P_n^i < \infty$. The proof for the other cases is exactly the same and so will not be given. The idea essentially is that during the outward loop $[T_i, S_i)$ the quantities $\omega_2(t)$ and $\omega_3(t)$ remain constant. So the only way that $\{\omega_1 = \omega_2 = \omega_3\}$ can occur within the outward loop is if for some i and j

(i) $\omega_2(t) = \omega_3(t) = i, \omega_1(t) = j$ and

(ii) either maxwind or minwind is greater than $|i-j|$.

By approximation (#) of Lyons and McKean,

$$P\{\omega_2(t) = \omega_3(t) = i \text{ at the start of the } (n+1)\text{th loop } |c_1(t), c_2(t), c_3(t)\}$$
$$\leq \frac{K}{\sqrt{c_2 c_3}} e^{-i^2/2c_2} e^{-i^2/2c_3}.$$

Given that $\omega_2(t) = \omega_3(t) = i$ and $\omega_1(t) = j$ at the beginning of the $(n+1)$th loop, we obtain from Lemma 1.1 that

$$P\{\omega_1 = \omega_2 = \omega_3 \text{ during the outward loop}\} \frac{K}{|i-j|+1}.$$

Therefore, given that $\omega_2 = \omega_3 = i$ and $(c_1(t), c_2(t), c_{3(t)})$ at the start of the outward loop, the probability that all three ω_k are equal during the outward loop is at most

$$K_1 \sum_{-\infty}^{\infty} \frac{1}{\sqrt{c_1}} e^{-j^2/2c_1} \frac{1}{|i-j|+1}.$$

Using approximations similar to those in §1, we find that this is less than

$$K_2 \left\{ \frac{\log(|i|+1)}{\sqrt{c_1}} + \frac{1}{|i|+1} \right\}.$$

So the probability that $\{\omega_1 = \omega_2 = \omega_3\}$ during the $(n+1)$th outward loop, given that the numbers of crossings at the start of the loop are c_1, c_2, c_3 respectively, is at most

$$\begin{aligned} &K_3 \sum_{i=-\infty}^{\infty} \frac{1}{\sqrt{c_2 c_3}} e^{-i^2/2c_2} e^{-i^2/2c_3} \left[\frac{\log(|i|+1)}{\sqrt{c_1}} + \frac{1}{|i|+1} \right] \\ &\leq K_4 \left[\frac{\log(c_3 c_2/(c_2+c_3))}{\sqrt{c_2+c_3}} \cdot \frac{1}{\sqrt{c_1}} + \frac{1}{\sqrt{c_2 c_3}} \right]. \end{aligned}$$

If $\min\{c_1, c_2, c_3\} \geq n^2/\log(n)^2$, then the above expression cannot be greater than $K_5((\log(n)^3)/n^2)$. We conclude that

$$P_n^1 \leq P\left\{ \min\{c_1, c_2, c_3\} < \frac{n^2}{\log(n)^2} \right\} + K_5 \frac{\log(n)^3}{n^2}$$

so

$$P_n^1 \leq 3K_1 e^{-h(\log(n)^2)} + 3K_2 e^{-hn} + K_3(\log(n)^3)/n^2.$$

Therefore $\sum P_n^1 < \infty$ and the proof is completed. Q.E.D.

2. The result which has just been proven has a geometric consequence.

The covering surface of $C\backslash\{1,-1\}$ can be thought of as the upper half plane, H. Each preimage of $C\backslash\{1,-1\} \cap \{z\colon \operatorname{im}(z) > 0\}$ is in a 1-1 correspondence with an element of the homotopy group of $C\backslash\{1,-1\}$. Transience of Brownian motion on H shows that the Brownian path in $C\backslash\{1,-1\}$ becomes more and more tangled up about the two deleted points as $t \to \infty$. See McKean (**1969**) for details, see also Durrett (**1983**) for a more direct proof of this fact.

Consider now a compact disc $D = \{z\colon |z-i| \leq 1/2\}$ in $C\backslash\{-1,1\}$. The preimages of D are also in a 1-1 correspondence with the elements of the homotopy group of $C\backslash\{1,-1\}$. Similarly there is a (not 1-1) correspondence between every preimage of D and a branch of $(\arg(z-1), \arg(z+1))$ on D such that for any branch B, the preimages of D corresponding to B are "dense" on the real line in the sense that

> for each $x \in R^1$ and each $\varepsilon > 0$ there exists a preimage of D corresponding to B in $\{z\colon |z-x| < \varepsilon\}$.

The results of Lyons and McKean (**1984**) and McKean and Sullivan (**1984**) are equivalent to the statement that given a branch B, Brownian motion in H eventually quits all the preimages of D corresponding to B. The result of proposition tells us in addition that Brownian motion in H a.s. cannot encircle preimages corresponding to B. By a result of Burgess Davis (**1979**), this means that for Lebesgue a.e. x on R^1 and for each $\alpha \in (0, \pi/2)$, there exists $\varepsilon(x)$ such that {preimages of D corresponding to $B\} \cap \{z\colon |z-x| < \varepsilon; |\arg(z-x) - \pi/2| < \alpha\}$ is empty.

REFERENCES

B. Davies (**1979**), *Brownian motion and analytic functions*, Ann. Prob. **7**, 913–321.

R. Durrett (**1984**), *Brownian motion and martingales in analysis*, Wadsworth, Monterey, Calif.

T. Lyons and H. P. McKean (**1984**), *Windings of the plane Brownian motion*, Adv. in Math. **51**, 212–225.

H. P. McKean (**1969**), *Stochastic integrals*, Academic Press, New York.

H. P. McKean and D. Sullivan (**1984**), *Brownian motion and harmonic functions on the class surface of the thrice punctured sphere*, Adv. in Math. **16**, 203–211.

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