PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 103, Number 3, July 1988

ON TWO PROBLEMS CONCERNING BAIRE SETS IN NORMAL SPACES

ZOLTAN BALOGH

(Communicated by Dennis Burke)

ABSTRACT. Two problems will be dealt with. The first problem, due to Katetov, asks whether there is a normal, nonperfect T_2 space X such that the Baire and Borel algebras in X coincide. The second problem, due to Ross and Stromberg, asks whether each closed Baire set has to be zero set in a normal, locally compact T_2 space. Several consistent examples of spaces satisfying the requirements of the first problem will be constructed. A counterexample to the second problem is given in ZFC.

Introduction. If X is a topological space, and A is a subset of X, then A is said to be a *Borel* (resp. *Baire*) subset of X, if A belongs to the σ-algebra generated by the closed sets (resp. the zero sets) of X. Note that in perfectly normal spaces the Borel and Baire σ-algebras coincide. Let us call a Hausdorff space X *Baire-ly perfect* if the Borel and the Baire σ-algebras of X coincide. Note that this is equivalent to the condition that each closed subset of X is a Baire set. Let us say that a Hausdorff space X is *Baire-ly perfectly normal* if X is Baire-ly perfect and normal. The following problem is due to M. Katetov [**Ka**, p. 74].

PROBLEM 1. Is there a Baire-ly perfectly normal, nonperfect space?

A related problem is the following question of Ross and Stromberg [**RS**, p. 152].

PROBLEM 2. If X is a normal, locally compact Hausdorff space and A is a closed Baire set in X, is A a zero set?

There have been several partial results concerning these problems as summarized in the following:

THEOREM. *Let X be a normal T_1 space, and let A be a closed Baire subset of X. Then A is a zero set in X if one of the following conditions hold.*

(1) *X is compact* [**H**, 51.D],
(2) *X is a paracompact, locally compact space* [**C**],
(3) *X is a submetacompact, locally compact space* [**Bu**],
(4) *X is Lindelöf and Čech-complete* [**C**],
(5) *X is a subparacompact $P(\omega)$-space* [**Ha**].

Thus, in the above five classes of spaces, the answer to Problem 1 is no. In the first three classes of spaces, the answer to Problem 2 is yes.

The aim of this paper is to give answers to these problems in general.

In §1, a Baire-ly perfectly normal, locally compact, locally countable, hereditarily separable, nonperfect space is constructed assuming the continuum hypothesis.

Received by the editors July 25, 1986 and, in revised form, January 5, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 54C50, 54D45; Secondary 54A35.
Key words and phrases. Baire versus Borel, Baire-ly perfectly normal, locally compact space.

Thus the answer to Problem 1 is consistently yes. Since our example is hereditarily separable and non-Lindelöf, some set-theoretic hypothesis is needed for its construction [**Sz**, **T**]. It is unknown to the author whether the answer to Problem 1 is yes in ZFC. Two more consistent examples of Baire-ly perfectly normal, nonperfect spaces are also given in §1. These spaces are easier to construct, but they are not locally compact.

In §2, a counterexample to Problem 2 is constructed in ZFC.

Our terminology and notation follows the standards of set theory and set-theoretic topology as is used in [**K**] and [**KV**], respectively. *Throughout the paper all spaces are assumed to be Hausdorff topological spaces.* Given a space X with topology τ, $\mathrm{cl}_X(A)$ or $\mathrm{cl}_\tau(A)$ will denote the closure of a subset A of X in (X,τ).

1. Baire-ly perfectly normal, nonperfect spaces.

PROPOSITION 1.1. *Let X be a Baire-ly perfect space, A be a Baire set in X, Y be a Baire subset of the subspace A. Then Y is a Baire subset of the space X, too.*

PROOF. It is enough to prove this for all closed Baire subsets Y of the subspace A. However, if Y is closed in A, then $Y = \mathrm{cl}_X(Y) \cap A$ is the intersection of two Baire subsets of X. Thus Y is a Baire subset of X.

PROPOSITION 1.2. *Let (X,τ) be a Baire-ly perfect space, A be a Baire set in (X,τ). Further, suppose that ρ is a topology on X satisfying the following properties:*

(a) *ρ is finer than τ and A is ρ-closed;*

(b) *each Borel subset of $(A,\rho \mid A)$ is a Baire set in $\tau \mid A$;*

(c) *each Borel subset of $(X-A,\rho \mid (X-A))$ is a Baire set in $\tau \mid (X-A)$.*

Then (X,ρ) is Baire-ly perfect.

PROOF. Let F be a ρ-closed subset of X. Then $F \cap A$ is a Borel subset of $(A,\rho \mid A)$. By (b), $F\cap A$ is a Baire subset of $(A,\tau \mid A)$. By Proposition 1.1, $F \cap A$ is a Baire subset of (X,τ), too.

A similar argument shows that $F\cap (X-A)$ is a Baire subset of (X,τ). Thus $F = (F\cap A)\cup (F\cap (X-A))$ is a Baire set in τ, and, a fortiori, it is a Baire set in ρ.

The following result is a modification of the "Kunen line" [**JKR**] for spaces X with a distinguished set A.

THEOREM 1.3 (CH). *Let (X,τ) be a hereditarily separable, first countable T_2 space of cardinality 2^ω, and let A be a nonvoid subset of X. Then there is a topology ρ on X with the following properties:*

(1) *ρ is finer than τ and A is ρ-closed;*

(2) *(X,ρ) is locally compact and locally countable;*

(3) *for every subset E of $X-A$, $\mathrm{cl}_\tau(E) - \mathrm{cl}_\rho(E)$ is countable;*

(4) *for every subset D of A, $(\mathrm{cl}_\tau(D) - \mathrm{cl}_\rho(D))\cap A$ is countable.*

PROOF. By CH, let $X = \{x_\alpha\colon \alpha\in\omega_1\}$, $[A]^\omega = \{D_\alpha\colon \alpha\in\omega_1\}$, $[X-A]^\omega = \{E_\alpha\colon \alpha\in\omega_1\}$. For each $\alpha\in\omega_1$, let $X_\alpha = \{x_\beta\colon \beta\in\alpha\}$. Without loss of generality we may assume that $D_\alpha\cup E_\alpha\subset X_\alpha$ for every $\alpha\in\omega_1$.

Let $\mathcal{D}_\alpha = \{D_\beta\colon \beta\in\alpha \text{ and } x_\alpha\in \mathrm{cl}_\tau(D_\beta)\}$, $\mathcal{E}_\alpha = \{E_\beta\colon \beta\in\alpha \text{ and } x_\alpha\in\mathrm{cl}_\tau(E_\beta)\}$ $(\alpha\in\omega_1)$.

Inductively, for each $\alpha \in \omega_1$, we shall construct a topology ρ_α on X_α such that the following conditions are satisfied.

(1α) ρ_α is finer than $\tau \mid X_\alpha$ and $A \cap X_\alpha$ is ρ_α-closed.

(2α) ρ_α is locally compact.

(3α) For every $\beta \in \alpha$, (X_β, ρ_β) is an open subspace of (X_α, ρ_α).

(4α) If $\alpha = \beta + 1$ and $x_\beta \in X - A$, then $x_\beta \in \mathrm{cl}_{\rho_\alpha}(E)$ for every $E \in \mathcal{E}_\beta$.

(5α) If $\alpha = \beta + 1$ and $x_\beta \in A$, then $x_\beta \in \mathrm{cl}_{\rho_\alpha}(E)$ and $x_\beta \in \mathrm{cl}_{\rho_\alpha}(D)$ for every $E \in \mathcal{E}_\beta$ and $D \in \mathcal{D}_\beta$.

Indeed, let $\rho_0 = \phi$. Next, suppose that $\alpha \in \omega_1 - \{\phi\}$, and for every $\gamma < \alpha$ we have already defined ρ_γ in such a way that (1γ)–(5γ) are satisfied. If α is a limit ordinal, then let ρ_α be the inductive limit topology of the spaces $\{(X_\gamma, \rho_\gamma) : \gamma \in \alpha\}$.

If $\alpha = \beta + 1$, then let $\mathcal{D}_\beta = \{D_n : n \in \omega\}$ and $\mathcal{E}_\beta = \{E_n : n \in \omega\}$ be enumerations of $\mathcal{D}_\beta$ and $\mathcal{E}_\beta$, respectively, in such a way that each member is listed infinitely many times. If $\mathcal{D}_\beta$ and $\mathcal{E}_\beta$ are both empty, then make x_β isolated in ρ_α. If not, then, since (X, τ) is first countable, there is a sequence $\{y_n : n \in \omega\} \subset X_\beta$ such that

(a) $\{y_n : n \in \omega\}$ converges to x_β in $\tau \mid X_\alpha$;

(b) if $x_\beta \in X - A$, then $y_n \in E_n$ for every $n \in \omega$;

(c) if $x_\beta \in A$, then $y_{2n} \in E_n$ and $y_{2n+1} \in D_n$ for each $n \in \omega$.

Since ρ_β refines $\tau \mid X_\beta$, $\{y_n : n \in \omega\}$ is closed discrete in (X_β, ρ_β). Since (X_β, ρ_β) is locally compact, countable and metrizable, there is a discrete family $\{C_n : n \in \omega\}$ of clopen compact subsets of X_β such that each τ-neighborhood of x_β contains all but finitely many of the C_n's, $y_n \in C_n$ for every $n \in \omega$, and, if $x_\beta \in X - A$, then $C_n \subset X - A$ for every $n \in \omega$. Let $\{G_n : n \in \omega\}$, where $G_n = \{x_\beta\} \cup \bigcup\{C_i : i \geq n\}$ ($n \in \omega$) be a neighborhood base for x_β in ρ_α. Neighborhood bases in ρ_α for points of X_β are the same as their neighborhood bases in ρ_β. It is easy to show that ρ_α satisfies conditions (1α)–(5α).

Finally, let ρ be the inductive limit of the topologies $\{\rho_\alpha : \alpha \in \omega\}$. We leave the routine verification that (X, ρ) satisfies the conditions of Theorem 1.3 to the reader.

REMARK. If in Theorem 1.3, we start with a second countable regular space (X, τ), and we let A be a Baire subset of X, then by Proposition 1.2, (X, ρ) will be a Baire-ly perfect space. If A is "complicated enough", then it will not be a G_δ set in (X, ρ), so (X, ρ) will not be perfect. The problem is, how to make (X, ρ) normal. To make sure the normality of (X, ρ), we shall make another use of CH to prepare the right input space (X, τ).

PROPOSITION 1.4 (CH). *If A is not an $F_{\sigma\delta}$ subset of the real line $\mathbf{R}$, then there is a subset P of $\mathbf{R} - A$ of cardinality 2^ω such that $P \cap Y$ is countable for all subsets Y of $\mathbf{R} - A$ which are G_δ sets in $\mathbf{R}$.*

PROOF. By CH, let $\{Y_\alpha : \alpha \in \omega_1\}$ enumerate all subsets of $\mathbf{R} - A$ which are G_δ sets in $\mathbf{R}$. Using transfinite induction, for each $\alpha \in \omega_1$ pick a point $x_\alpha \in (\mathbf{R} - A) - (\bigcup\{Y_\beta : \beta \in \alpha\} \cup \{x_\beta : \beta \in \alpha\})$. This is possible, since $\mathbf{R} - A$ is not a $G_{\delta\sigma}$ set in $\mathbf{R}$. Then $P = \{x_\alpha : \alpha \in \omega_1\}$ is as required.

THEOREM 1.5 (CH). *There is a Baire-ly perfectly normal, locally compact, locally countable space which is not perfectly normal.*

PROOF. Let A be a Baire, but not $F_{\sigma\delta}$ subset of $\mathbf{R}$, and let P be as in Proposition 1.4. Let $X = A \cup P$ and τ be the subspace topology on X inherited from $\mathbf{R}$. Finally, let ρ be the topology on X satisfying (1)–(4) of Theorem 1.3. We shall prove that (X,ρ) is a space the existence of which was claimed in our theorem.

First of all, by (1) and (2), (X,ρ) is a locally compact, locally countable T_2 space.

CLAIM. Every ρ-closed subset F of P is countable.

Indeed, let $\Delta = \mathrm{cl}_\tau(F) - F$ and $Y = \mathrm{cl}_{\mathbf{R}}(F) - \Delta$. By (3), Δ is countable, so Y is a G_δ set in $\mathbf{R}$. Thus $F = X \cap Y = P \cap Y$ is countable.

By the Claim, (X,ρ) is not perfect(ly normal).

By (3), each closed subset in $(P,\rho \mid P)$ is G_δ in $(P,\tau \mid P)$. Thus each Borel subset of $(P,\rho \mid P)$ is a Baire subset of $(P,\tau \mid P)$ so condition (c) of Proposition 1.2 is satisfied. By (4), condition (b) of Proposition 1.2 is satisfied. By Proposition 1.2, (X,ρ) is Baire-ly perfect.

In order to show that (X,ρ) is normal, it is enough to show that given two distinct ρ-closed subsets F_1 and F_2 of X, there is a countable family $\mathcal{G}$ of ρ-open sets such that

(i) $\bigcup \mathcal{G} \supset F_1$;

(ii) for every $G \in \mathcal{G}$, $\mathrm{cl}_\rho(G) \cap F_2 = \varnothing$.

To see this, let $A_1 = F_1 \cap A$, $P_1 = F_1 \cap P$, $\Delta_1 = \mathrm{cl}_\tau(F_2) \cap A_1$. Since $(A_1 - \Delta_1) \cap \mathrm{cl}_{\mathbf{R}}(F_2) = \varnothing$ and the natural topology of $\mathbf{R}$ is second countable, there is a countable family $\mathcal{U}$ of open subsets of $\mathbf{R}$ such that

(i′) $\bigcup \mathcal{U} \supset A_1 - \Delta_1$;

(ii′) For every $U \in \mathcal{U}$, $\mathrm{cl}_{\mathbf{R}}(U) \cap F_2 = \varnothing$.

Now, let $P_1' = P_1 - \bigcup \mathcal{U}$, $\Delta_1' = \mathrm{cl}_\tau(P_1') - F_1$, $Y = \mathrm{cl}_{\mathbf{R}}(P_1') - (\bigcup \mathcal{U} \cup \Delta_1 \cup \Delta_1')$. Since $\Delta_1' \subset \mathrm{cl}_\tau(P_1') - \mathrm{cl}_\rho(P_1')$, Δ_1' is countable. Thus Y is G_δ set in $\mathbf{R}$. Therefore $P_1' = Y \cap P$ is countable. Let $\Delta = \Delta_1 \cup P_1'$. For each $x \in \Delta$, let $G(x) \ni x$ be a ρ-open set such that $\mathrm{cl}_\rho(G(x)) \cap F_2 = \varnothing$. Then $\mathcal{G} = \{G(x) : x \in \Delta\} \cup \{U \cap X : U \in \mathcal{U}\}$ satisfies (i) and (ii).

If we drop "locally compact", then consistent counterexamples to Katetov's question are easier to construct. Two such examples are given below.

THEOREM 1.6 (CH). *There is a Lindelöf, Baire-ly perfectly normal, hereditarily separable, nonperfect space.*

PROOF. Let Q denote the set of all rationals in $\mathbf{R}$. Since CH holds, there is an uncountable subset P of $\mathbf{R} - Q$ in such a way that $P \cap Y$ is countable for all subsets Y of $\mathbf{R} - Q$ which are closed sets in $\mathbf{R}$. (Cf. the proof of Proposition 1.4.) Let τ denote the subspace topology on P inherited from the natural topology of $\mathbf{R}$. Apply the machine of Theorem 1.3 with $X = P$ and $A = \phi$ to get a refinement ρ of τ on P. Let σ denote the topology on $X = Q \cup P$ determined by the following two conditions:

(1) if $x \in Q$, then $\{G_n : n \in \omega\}$, where $G_n = (x - 1/(n+1), x + 1/(n+1)) \cap X$ $(n \in \omega)$, is an open σ-neighborhood base for X,

(2) (P,ρ) is an open subspace of (X,σ).

Since ρ is locally compact and zerodimensional, it follows that (X,σ) is a regular T_1 space. By the construction of P, (X,σ) also is Lindelöf and thus it is normal. By Proposition 1.2, it is Baire-ly perfect. However, Q is not a G_δ set in (X,σ), so (X,σ) is not perfect.

The last example makes use of the following result of A. Miller.

THEOREM 1.7 [**Mi**]. *There is a model of set theory in which there is a subspace M of the real line with the following properties.*

(a) *Every subset of M is a Baire set in the subspace topology of M.*

(b) *There is a subset $A \subset M$ such that A is not a G_δ set in M.*

EXAMPLE 1.8. In Miller's model, consider his space M with a subset A as above. Define a new topology ρ on M by the following two conditions.

(1) For every $x \in A$, $\{G_n : n \in \omega\}$, where $G_n = (x - 1/(n+1), x + 1/(n+1)) \cap M$ is a neighborhood base for x.

(2) All points of $M - A$ are isolated in ρ.

The space (M, ρ) obtained in this way could be called the Miller-Michael Line [**M**]. As the original space of Michael [**M**], it is hereditarily paracompact, and thus normal. Since ρ refines the natural topology of M, every subset in (M, ρ) is Baire and Borel. Thus (M, ρ) is Baire-ly perfectly normal. Since we left intact the natural neighborhoods of points in A, A is not a G_δ subset in (M, ρ). Thus (M, ρ) is not perfect.

2. On closed Baire sets in locally compact, normal spaces. The following construction makes use of the technique of [**vDW**] and [**W**].

THEOREM 2.1. *Let A be a nonvoid Baire subset of the real line* $\mathbf{R}$. *Then there is a topology ρ on* $\mathbf{R}$ *with the following properties.*

(1) *ρ is finer than the natural topology τ on* $\mathbf{R}$;

(2) *each point of* $\mathbf{R} - A$ *is isolated in ρ*;

(3) *ρ is locally compact and locally countable*;

(4) *if $D \subset A$ and $E \subset \mathbf{R}$ are disjoint ρ-closed sets then $\mathrm{cl}_\tau(E) \cap D$ is countable*;

(5) *ρ is normal.*

PROOF. Let $\{\langle D_\alpha, E_\alpha\rangle : \alpha \in 2^\omega\}$ enumerate all pairs of countable subsets of $\mathbf{R}$ such that

(a) $D_\alpha \subset A$,

(b) $\Delta_\alpha = \mathrm{cl}_\tau(E_\alpha) \cap \mathrm{cl}_\tau(D_\alpha) \cap A$ is uncountable.

Without loss of generality we may assume that each pair was repeated 2^ω times. Inductively, for each $\alpha < 2^\omega$, we shall construct a space (X_α, ρ_α) such that the following conditions are satisfied:

(1α) ρ_α is finer than $\tau \mid X_\alpha$;

(2α) $A \cap X_\alpha$ is closed in ρ_α;

(3α) ρ_α is locally compact and locally countable;

(4α) for every $\beta \in \alpha$, (X_β, ρ_β) is an open subspace of (X_α, ρ_α);

(5α) if $\alpha = \beta + 1$, and there is a $\gamma \in \beta$ with $(D_\beta, E_\beta) = (D_\gamma, E_\gamma)$, then $X_{\beta+1} - X_\beta$ contains a point $x_\beta \in \Delta_\beta$ such that $x_\beta \in \mathrm{cl}_{\rho_\alpha}(E_\beta) \cap \mathrm{cl}_{\rho_\alpha}(D_\beta) \cap A$.

Indeed, let X_0 be the set of rationals Q and ρ_0 be the discrete topology on X_0. Next, suppose that $0 \neq \alpha < 2^\omega$, and for every $\gamma \in \alpha$ we have already defined (X_γ, ρ_γ) in such a way that (1γ)–(5γ) are satisfied. If α is a limit ordinal, then let ρ_α be the inductive limit of the spaces $\{(X_\gamma, \rho_\gamma) : \gamma \in \alpha\}$.

If $\alpha = \beta + 1$, then let us choose x_β to be an arbitrary point from $\Delta_\beta - X_\beta$. Note that $\Delta_\beta - X_\beta$ is nonvoid. Indeed, since Δ_β is an uncountable Borel subset in τ, it has cardinality 2^ω, whereas X_β has cardinality $< 2^\omega$. There are two cases to consider.

Case 1. Assume that $D_\beta \cup E_\beta \subset X_\beta$. Then choose a sequence $\{y_n : n \in \omega\} \subset X_\beta$ such that

(a) $\{y_n : n \in \omega\}$ converges to x_β in τ,

(b) $y_n \in D_\beta$ for each even $n \in \omega$,

(c) $y_n \in E_\beta$ for each odd $n \in \omega$.

Since ρ_β refines $\tau \mid X_\beta$, $\{y_n : n \in \omega\}$ is closed discrete in (X_β, ρ_β). Since (X_β, ρ_β) is locally compact and locally countable (and thus, zero-dimensional), there is a discrete family $\{C_n : n \in \omega\}$ of countable clopen compact subsets of (X_β, ρ_β) converging to x_β in τ such that $y_n \in C_n$ for every $n \in \omega$.

Let $X_\alpha = X_{\beta+1} = X_\beta \cup \{x_\beta\}$. Let $\{G_n : n \in \omega\}$, where

$$G_n = \{x_\beta\} \cup \bigcup\{C_i : i \geq n\} \qquad (n \in \omega),$$

be a neighborhood base for x_β in ρ_α. Neighborhood bases in ρ_α for points of X_β are the same as their neighborhood bases in ρ_β.

Case 2. Assume that $D_\beta \cup E_\beta \not\subset X_\beta$. Then let $X_{\beta+1} = X_\beta \cup D_\beta \cup E_\beta$, and let $\rho_{\beta+1}$ be the topology on $X_{\beta+1}$ determined by the following two conditions.

(i) (X_β, ρ_β) is a clopen subspace of $(X_{\beta+1}, \rho_{\beta+1})$;

(ii) each point of $X_{\beta+1} - X_\beta$ is isolated in $\rho_{\beta+1}$.

It is easy to see that the space (X_α, ρ_α) defined above satisfies (1α)–(5α) in each case.

Finally, let ρ be the topology on $\mathbf{R}$ defined by the following two conditions.

(A) If $x \in X_\alpha$ for some ordinal $\alpha < 2^\omega$, then let a neighborhood base for x in ρ be the same as that in ρ_α,

(B) If $x \notin \bigcup\{X_\alpha : \alpha < 2^\omega\}$, then let x be isolated in ρ.

ρ clearly satisfies (1), (2) and (3). To see that it also satisfies (4), let $D \subset A$, $E \subset \mathbf{R}$ be disjoint ρ-closed sets. Assume indirectly that $\Delta = \mathrm{cl}_\tau(E) \cap D$ is uncountable. Then there is a pair $\langle D', E'\rangle$ of countable sets such that D' is τ-dense in D, and E' is τ-dense in E. Since $\mathrm{cl}_\tau(E') \cap (\mathrm{cl}_\tau(D') \cap A) \supset \Delta$, $\langle D', E'\rangle = \langle D_\alpha, E_\alpha\rangle$ for 2^ω many $\alpha < 2^\omega$. By (5α) this implies that $|\mathrm{cl}_\rho(E_\alpha) \cap (\mathrm{cl}_\rho(D_\alpha) \cap A)| = 2^\omega$, in contradiction with $\mathrm{cl}_\rho(E_\alpha) \cap (\mathrm{cl}_\rho(D_\alpha) \cap A) \subset E \cap D = \varnothing$.

To see that $(\mathbf{R}, \rho)$ is normal, it is enough to prove that for any pair F_1, F_2 of disjoint ρ-closed sets, there is a countable ρ-open cover $\mathcal{G}$ of F_1 such that for each $G \in \mathcal{G}$, $\mathrm{cl}_\rho(G) \cap F_2 = \varnothing$. To see this, let $A_1 = F_1 \cap A$ and $\Delta_1 = \mathrm{cl}_\tau(F_2) \cap A_1$. Since τ has a countable base there is a countable family of $\mathcal{U}$ of τ-open subsets of $\mathbf{R}$ such that

(*) $\bigcup \mathcal{U} \supset A_1 - \Delta_1$;

(**) for every $U \in \mathcal{U}$, $\mathrm{cl}_\tau(U) \cap F_2 = \varnothing$.

By (4), Δ_1 is countable. For each $x \in \Delta_1$, let $G(x) \ni x$ be a ρ-open set such that $\mathrm{cl}_\rho(G(x)) \cap F_2 = \varnothing$. Then $\mathcal{G} = \{G(x) : x \in \Delta\} \cup \mathcal{U} \cup \{F_1 - A\}$ is as required.

COROLLARY 2.2. *There is normal, locally compact, locally countable space with a closed Baire subset which is not a zero-set.*

PROOF. Let A be a Baire but not $G_{\delta\sigma}$ subset A of $\mathbf{R}$. Apply Theorem 2.1 to obtain a new topology on $\mathbf{R}$ such as described there. Then the space $(\mathbf{R}, \rho)$ is normal, locally compact and locally countable. Since ρ refines τ, A is Baire set in $(\mathbf{R}, \rho)$, too. However, A is not a G_δ set in $(\mathbf{R}, \rho)$. Indeed, making use of (4) of Theorem 2.1 with $D = A$, it follows that if A was a G_δ subset in $(\mathbf{R}, \rho)$, then A would be a $G_{\delta\sigma}$ subset in the natural topology τ of $\mathbf{R}$.

REFERENCES

[**BB**] Z. Balogh and H. Bennett, *On two classes of sets containing all Baire sets and all co-analytic sets*, Topology Appl. (to appear).

[**Bu**] D. Burke, Lectures given at Texas Tech University, February 1986, and at the Spring Topology Conference, Lafayette, La., April 1986.

[**C**] W. W. Comfort, *Closed Baire sets are (sometimes) zero-sets*, Proc. Amer. Math. Soc. **25** (1970), 870–875.

[**vDW**] E. van Douwen and H. Wicke, *A real, weird topology on the reals*, Houston J. Math. **3** (1977), 141–151.

[**H**] P. R. Halmos, *Measure theory*, Van Nostrand, New York, 1950.

[**Ha**] R. W. Hansell, *Topological properties of Souslin subsets*, Trans. Amer. Math. Soc. **293** (1986), 613–622.

[**JKR**] I. Juhasz, K. Kunen and M. E. Rudin, *Two more hereditarily separable non-Lindelöf spaces*, Canad. J. Math. **28** (1976), 998–1005.

[**K**] K. Kunen, *Set theory*, North-Holland, Amsterdam, New York and Oxford, 1980.

[**Ka**] M. Katetov, *Measures in fully normal spaces*, Fund. Math. **38** (1951), 73–84.

[**KV**] K. Kunen and F. E. Vaughan (eds.), *Handbook of set-theoretic topology*, North-Holland, Amsterdam, 1984.

[**M**] E. Michael, *The product of a normal space and a metric space need not be normal*, Bull. Amer. Math. Soc. **69** (1963), 375–376.

[**Mi**] W. Miller, *On the length of Borel hierarchies*, Ann. Math. Logic **16** (1979), 233–267.

[**RS**] K. A. Ross and K. Stromberg, *Baire sets and Baire measures*, Ark. Mat. **6** (1965), 151–160.

[**T**] S. Todorcevic, *No S spaces*, manuscript.

[**Sz**] Z. Szentmiklóssy, *S-spaces and L-spaces under Martin's axiom*, Colloq. Math. Soc. János Bolyai 23 (Budapest 1978), II, North-Holland, Amsterdam, 1980, pp. 1139–1145.

[**W**] W. Weiss, *A countably paracompact nonnormal space*, Proc. Amer. Math. Soc. **79** (1980), 487–490.

DEPARTMENT OF MATHEMATICS, TEXAS TECH UNIVERSITY, LUBBOCK, TEXAS 79409

DEPARTMENT OF MATHEMATICS, KOSSUTH UNIVERSITY, DEBRECEN, HUNGARY (Current address)