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FIXED POINT THEOREMS FOR COMPACT ACYCLIC METRIC SPACES

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ABSTRACT. Multifunctions $F: X \to 2^M$ are studied where $X \subset M$ is either a compact acyclic finite-dimensional ANR or a compact acyclic lc subspace. Conditions are found for the existence of v such that $d(v, Fv) = d(X, Fv)$ and, also, for the existence of fixed points.

1. Introduction and preliminaries. In [**7**], McClendon proved variations of Ky Fan's minimax inequality in [**5**]. His main idea is the weakening of "convex" to "contractible" or "acyclic". He used his results to obtain some variational inequalities and fixed point theorems on a function $f: X \to (M, d)$ with a condition on the "boundary", where $X \subset M$ is either a compact acyclic finite-dimensional ANR or a compact acyclic lc subspace.

In the present paper, we generalize these fixed point theorems to a multifunction $F: X \to 2^M$ with a more general condition than the one on the "boundary". Actually, we give a number of equivalent formulations of such fixed point theorems in general setting.

We begin with the following two results of McClendon [**7**, Theorems 3.1 and 3.2].

THEOREM 1.1 [**7**, THEOREM 3.1]. *Suppose that X is a compact acyclic finite-dimensional ANR. Suppose $p: X \times X \to R$ is a function such that*

(a) *$\{(x,y) | p(x,x) > p(y,x)\}$ is open in $X \times X$, and*

(b) *$\{y | p(x,x) > p(y,x)\}$ is contractible or empty for all $x \in X$.*

Then there exists a point $v \in X$ such that $p(v,v) \leq p(w,v)$ for all $w \in X$.

THEOREM 1.2 [**7**, THEOREM 3.2]. *Suppose that X is a compact acyclic lc space. Let $p: X \times X \to R$ be a function such that*

(a) *p is continuous on the diagonal $\{(x,x) | x \in X\}$,*

(b) *p is l.s.c.,*

(c) *$p(x, \cdot)$ is u.s.c. for all $x \in X$, and*

(d) *$\{x | p(x,y) \leq s\}$ is acyclic or empty for all $s \in R$ and $y \in X$.*

Then there exists a point $v \in X$ such that $p(v,v) \leq p(w,v)$ for all $w \in X$.

REMARK. Due to McClendon's proof of Theorem 2.2 [**7**] it suffices to assume (d) for $s = p(y,y) - \delta(n)$, all n, where $\{\delta(n)\}$ is a sequence of positive reals converging to 0.

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A space is *contractible* if the identity map is homotopic to a constant, e.g., a nonempty convex set in a topological vector space (t.v.s.). A nonempty space is *acyclic* if it is connected and its Čech homology (with a fixed coefficient field) is zero in dimensions greater than zero. Note that every contractible space is acyclic, but not conversely.

A multifunction $F\colon X \to 2^Y$ is *continuous* if it is u.s.c. and l.s.c. ANR means ANR (metric). For the definition of a *lc space*, see [**1**, **8**, **9**]. Note that an ANR is an lc space and a finite union of compact convex subsets of a locally convex t.v.s. is an lc space. Let us denote $d(y, Fx) = \inf\{d(y,z) | z \in Fx\}$.

We add one more variational inequality.

THEOREM 1.3. *Let X be a compact acyclic lc space. Let $p\colon X \times X \to R$ be a continuous function such that $\{x | p(x,y) = P(y)\}$ is acyclic for all $y \in X$, where $P(y) = \inf\{p(x,y) | x \in X\}$. Then there exists a point $v \in X$ such that $p(v,v) \leq p(w,v)$ for all $w \in X$.*

PROOF. Define a multifunction $F\colon X \to 2^X$ by $F(y) = \{x | p(x,y) = P(y)\}$ for each $y \in X$. Since F is a closed-graph acyclic-valued multifunction, F has a fixed point $v \in X$ by Begle's result [**1**], and v is a desired one.

2. Main results. First from Theorem 1.1 we have

THEOREM 2.1. *Let X be a compact acyclic finite-dimensional ANR subset of a metric space (M,d), and $F\colon X \to 2^M \backslash \{\phi\}$ a multifunction satisfying*

(a) *$\{(x,y) | d(x,Fx) > d(y,Fx)\}$ is open in $X \times X$, and*

(b) *for each $x \in X$, $\{y \in X | d(x,Fx) > d(y,Fx)\}$ is contractible or empty.*

Then the following equivalent statements hold.

(i) *There exists a point $v \in X$ such that*

$$d(v,Fv) \leq d(w,Fv) \quad \text{for all } w \in X.$$

(ii) *If $T\colon X \to 2^M$ is a multifunction such that for each $x \in X \backslash Tx$ there exists a $y \in X$ such that $d(x,Fx) > d(y,Fx)$, T has a fixed point $v \in X$, that is, $v \in Tv$.*

(iii) *If $g\colon X \to M$ is a function such that for each $x \in X$ with $x \neq gx$, there exists a $y \in X$ such that $d(x,Fx) > d(y,Fx)$, then g has a fixed point $v \in X$.*

(iv) *If $T\colon X \to 2^X \backslash \{\phi\}$ is a multifunction such that $d(x,Fx) > d(y,Fx)$ holds for any $x \in X$ and $y \in Tx \backslash \{x\}$, then T has a stationary point $v \in X$, that is, $\{v\} = Tv$.*

(v) *If $\mathscr{F}$ is a nonempty family of functions $g\colon X \to X$ such that $d(x,Fx) > d(gx,Fx)$ holds for any $x \in X$ with $x \neq gx$, then $\mathscr{F}$ has a common fixed point $v \in X$.*

(vi) *If $g\colon X \to X$ is a function such that $d(x,Fx) > d(gx,Fx)$ holds for any $x \in X$ with $x \neq gx$, then g has a fixed point $v \in X$.*

PROOF. (i) Define $p\colon X \times X \to R$ by $p(x,y) = d(x,Fy)$. Then (i) follows from Theorem 1.1.

(i) $\Rightarrow$ (ii) Suppose $v \notin Tv$. Then there exists a $y \in X$ such that $d(v,Fv) > d(y,Fv)$, which contradicts (i).

(ii) $\Rightarrow$ (iii) Clear.

(iii) $\Rightarrow$ (iv) Suppose T has no stationary point, that is, $Tx \backslash \{x\} \neq \varnothing$ for each $x \in X$. Choose a choice function g on $\{Tx \backslash \{x\} | x \in X\}$. Then g has no fixed point. However, for any $x \in X$, we have $x \neq gx$ and there exists a $y \in Tx \backslash \{x\}$, say $y = gx$, such that $d(x, Fx) > d(y, Fx)$. Therefore, by (iii), g has a fixed point, a contradiction.

(iv) $\Rightarrow$ (v) Define a multifunction $T: X \to 2^X$ by $Tx := \{gx | g \in \mathscr{F}\} \neq \varnothing$ for all $x \in X$. Since $d(x, Fx) > d(gx, Fx)$ for any $x \in X$ and any $g \in \mathscr{F}$, by (iv), T has a stationary point $v \in X$, which is a common fixed point of $\mathscr{F}$.

(v) $\Rightarrow$ (vi) Put $\mathscr{F} = \{g\}$.

(vi) $\Rightarrow$ (i) Suppose that for any $x \in X$, there exists a $y \in X$ satisfying $d(x, Fx) > d(y, Fx)$. Choose gx to be one of such y. Then $g: X \to X$ has no fixed point by its definition. However, $d(x, Fx) > d(gx, Fx)$ for all $x \in X$. By (vi), g has a fixed point, a contradiction.

This completes our proof.

Define $\partial X = \partial_M X = \{y \in X |$ there is a $z \in M \backslash X$ with $d(z, y) = d(z, X)\}$ [**7**].

COROLLARY 2.1.1. *Let X be a compact acyclic finite-dimensional ANR subset of a metric space M, and $F: X \to 2^M \backslash \{\phi\}$ continuous compact-valued such that for all $x \in X$, $\{y \in X | d(x, Fx) > d(y, Fx)\}$ is contractible or empty.*

Then (i) $\sim$ (vi) *of Theorem* 2.1 *hold. Further, F has a fixed point if one of the following conditions holds.*

(1) *For each $x \in X \backslash Fx$, there exists a $y \in X$ such that $d(x, Fx) > d(y, Fx)$.*

(2) *$Fx \cap X \neq \varnothing$ for each $x \in \partial X$.*

(3) *$Fx \cap X \neq \varnothing$ for each $x \in X$.*

(4) *$Fx \subset X$ for each $x \in X$.*

PROOF. Since F is continuous, the condition (a) in Theorem 2.1 holds by Theorems 1 and 2 in Berge [**2**, p. 121]. Therefore, (i) $\sim$ (vi) hold.

(1) By putting $F = T$ in (ii), if condition (1) holds, then F has a fixed point.

(2) We show that (2) $\Rightarrow$ (1). Suppose that there exists an $x \in X \backslash Fx$ such that $d(x, Fx) \leq d(y, Fx)$ holds for all $y \in X$. If $Fx \cap X \neq \varnothing$, then $d(y, Fx) = 0$ for some $y \in Fx \cap X$. Therefore, $d(x, Fx) = 0$, whence $x \in Fx$, for Fx is compact, a contradiction. If $Fx \subset M \backslash X$, then there exists a $z \in Fx$ such that $d(x, z) = d(x, Fx)$ since Fx is compact. Therefore, $d(z, x) = d(x, Fx) \leq d(y, Fx) \leq d(y, z) = d(z, y)$ for all $y \in X$, that is, $d(z, x) = d(z, X)$. Hence $x \in \partial X$, and by (2), $Fx \cap X \neq \varnothing$, another contradiction.

Clearly, (4) $\Rightarrow$ (3) $\Rightarrow$ (2). This completes our proof.

REMARK. Corollary 2.1.1(4) is comparable to a well-known result that, if X is a compact acyclic ANR and $F: X \to 2^X \backslash \{\phi\}$ u.s.c. with Fx acyclic for each $x \in X$, then F has a fixed point (Eilenberg and Montgomery [**3**]). However, the following example of Gwinner [**6**, p. 575] shows that the lower semicontinuity of F in Corollary 2.1.1(1) is not dispensable.

EXAMPLE. Let $X = \{(c, 0) | c \in [0, 1]\} \subset R^2 = M$. Define $F: X \to 2^M \backslash \{\phi\}$ by

$$
\begin{aligned}
F(c, 0) &= \operatorname{co}\{(1, 1), (1, 2)\} \quad \text{if } 0 \leq c < 1, \\
F(1, 0) &= \operatorname{co}\{(0, 0), (1, 1), (1, 2)\}.
\end{aligned}
$$

Then F satisfies the hypothesis of Corollary 2.1.1(1) except the l.s.c. of F. However, F has no fixed point.

COROLLARY 2.1.2. *Let X be a compact acyclic finite-dimensional ANR subset of a metric space, and $f: X \to M$ continuous such that for all $x \in X$, the set $\{y \in X | d(x, fx) > d(y, fx)\}$ is contractible or empty.*

Then (i) $\sim$ (vi) *of Theorem* 2.1 *with $F = f$ hold. Further, f has a fixed point if one of the following conditions holds.*

(1) *For each $x \in X$ with $x \neq fx$, there exists a $y \in X$ such that $d(x, fx) > d(y, fx)$.*

(2) $f(\partial X) \subset X$.

(3) $f(X) \subset X$.

REMARK. Case (2) in Corollary 2.1.2 is due to McClendon [**7**, Theorem 3.6]. The case (3) follows from the well-known fact that every compact acyclic ANR has the fixed point property.

From Theorem 1.2, we have

THEOREM 2.2. *Let X be a compact acyclic lc subspace of a metric space (M, d), and $F: X \to 2^M \backslash \{\phi\}$ a multifunction satisfying*

(a) *$x \mapsto d(x, Fx)$ is continuous on X,*

(b) *$(x, y) \mapsto d(x, Fy)$ is l.s.c.,*

(c) *for each $x \in X$, $y \mapsto d(x, Fy)$ is u.s.c., and*

(d) *$\{x \in X | d(x, Fy) \leq t\}$ is acyclic or empty for all $y \in X$ and $t \in R$.*

Then the statements (i) $\sim$ (vi) *of Theorem* 2.1 *hold.*

PROOF. Define $p: X \times X \to R$ by $p(x, y) = d(x, Fy)$. Then (i) follows from Theorem 1.2. The equivalency of (i) $\sim$ (vi) is the same as Theorem 2.1.

REMARK. If $Fy \subset X$ and Fy is closed, then (d) implies that Fy is acyclic, for $Fy = \{x \in X | d(x, Fy) \leq 0\}$.

COROLLARY 2.2.1. *Let X be a compact acyclic lc subspace of a metric space M, and $F: X \to 2^M \backslash \{\phi\}$ continuous compact-valued such that $\{x \in X | d(x, Fy) \leq t\}$ is acyclic or empty for all $y \in X$ and $t \in R$. Then the conclusion of Corollary* 2.1.1 *holds.*

PROOF. For a continuous compact-valued multifunction F, the conditions (a), (b), and (c) of Theorem 2.2 hold, by Theorems 1 and 2 of Berge [**2**, p. 121].

REMARK. The case (4) is a consequence of Begle's well-known theorem in [**1**]: If X is a compact acyclic lc space and $\mathscr{C}(X)$ the class of all closed acyclic subsets of X, then any u.s.c. $F: X \to \mathscr{C}(X)$ has a fixed point. However, as in the remark following Corollary 2.1.1, Gwinner's example shows that the lower semicontinuity of F in Corollary 2.2.1 (1) is not dispensable.

COROLLARY 2.2.2. *Let X be a compact acyclic lc subspace of a metric space M, and $f: X \to M$ continuous such that $\{x \in X | d(x, fy) \leq t\}$ is acyclic for all $y \in X$ and $t \in R$. Then the conclusion of Corollary* 2.1.2 *holds.*

REMARK. Corollary 2.2.2 (i) and the case (2) are due to McClendon [**7**, Theorem 3.7 and Corollary 3.8]. Corollary 2.2.2 (i) generalizes Fan [**4**, Theorem 2]. For the case (3), if f is a self-map of X, then the condition on $\{x \in X | d(x, fy) \leq t\}$ is superfluous, by Begle's theorem mentioned above.

From Theorem 1.3, we have the following theorem as a final conclusion. The proof is parallel and some corollaries of it also can be obtained as in Theorems 2.1 and 2.2, so we omit them.

THEOREM 2.3. *Let X be a compact acyclic lc subspace of a metric space (M, d), and $F\colon X \to 2^M \backslash \{\phi\}$ a multifunction satisfying*

(a) *$d(x, Fy)$ is continuous on $X \times X$,*

(b) *$\{x | d(x, Fy) = \inf\{d(z, Fy) | z \in X\}\}$ is acyclic for each $y \in X$.*

Then statements (i) $\sim$ (vi) *of Theorem* 2.1 *hold.*

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