PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 103, Number 3, July 1988

EXTENSION OF RANDOM CONTRACTIONS

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(Communicated by R. Daniel Mauldin)

ABSTRACT. Let Ω be a measurable space. Let X and Y be separable Hilbert spaces and let D be a subset of X. Then every random contraction $f\colon \Omega \times D \to Y$ can be extended to a random contraction defined on all $\Omega \times X$. This statement remains true if Ω is a complete measurable space, X and Y are separable metric spaces and the pair (X, Y) has the Kirszbraun intersection property.

1. Introduction and notions. Let $\{X, d_1\}$ and $\{Y, d_2\}$ be metric spaces and let $f\colon D \to Y$ be a contraction from a nonempty subset D of X into Y, that is,

$$d_2(f(x_1), f(x_2)) \leq d_1(x_1, x_2) \qquad (x_1, x_2 \in X).$$

The extension problem for contractions asks under what conditions can one always guarantee the existence of a contraction $\tilde{f}\colon X \to Y$ such that $\tilde{f}|_D = f$. This problem has been studied by several authors; for a survey of basic results, see [**3** and **8**]. In particular, F. Valentine [**9**] proved that extension is always possible for a given pair (X, Y) if and only if the following property holds:

KIRSZBRAUN INTERSECTION PROPERTY. *For all choices $x_i \in X$, $y_i \in Y$ and $r_i > 0$, $i \in J$ (J an arbitrary index set) such that the intersection of the balls $S(x_j, r_j)$ in X is nonempty and $d_2(y_i, y_j) \leq d_1(x_i, x_j)$, $i, j \in J$, it follows that the intersection of the balls $S(y_i, r_i)$ in Y is also nonempty.*

Here $S(x, r)$ denotes the closed ball about x of radius r. Note that the Kirszbraun property (property (K)) is satisfied if X and Y are Hilbert spaces (see [**3**, **9**]). In this paper we are concerned with the extension problem for random contractions.

Let Ω be a measurable space with σ-algebra $\mathscr{A}$ and let D be a nonempty subset of X. A map $f\colon \Omega \times D \to Y$ is called a *random contraction* if, for every $x \in D$, $f(\cdot, x)$ is measurable and, for every $\omega \in \Omega$, $f(\omega, \cdot)$ is a contraction. Our major results state conditions under which there exists a random contraction $\tilde{f}\colon \Omega \times X \to Y$ such that $\tilde{f}|_{\Omega \times D} = f$.

Ideas from the theory of measurable multifunctions play a dominant role in the proofs. Let 2^X denote the family of all nonempty subsets of X. A map $\Gamma\colon \Omega \to 2^X$ is said to be *measurable* (resp. *weakly measurable*) if for every closed (resp. open) subset A of X, the set $\{\omega \in \Omega\colon \Gamma(\omega) \cap A \neq \varnothing\}$ belongs to $\mathscr{A}$. Note that if X is a metric space, measurability implies weak measurability. If, in addition, X is σ-compact, then measurability is equivalent to weak measurability. By a *measurable*

Received by the editors April 13, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 54C20; Secondary 47H09.
Key words and phrases. Random contraction, random isometry, extension, multifunction, measurable selection, separable metric space, Hilbert space.

selection of Γ we mean a measurable function $\gamma\colon \Omega \to X$ such that $\gamma(\omega) \in \Gamma(\omega)$ for every $\omega \in \Omega$.

2. Results.

LEMMA 1. *Let Ω be a measurable space and let X be a separable Hilbert space. Let $f_i\colon \Omega \to X$ and $\rho_i\colon \Omega \to [0, +\infty)$, $i = 1, 2, \ldots$, be measurable functions such that for every $\omega \in \Omega$,*

$$\Sigma(\omega) = \bigcap_{i \geq 1} S(f_i(\omega)\rho_i(\omega)) \neq \varnothing. \tag{1}$$

Then the multifunction $\Gamma\colon \Omega \to 2^X$ given by (1) *is weakly measurable.*

PROOF. By [**1**, Theorem III.4.1] the multifunctions $S(f_i(\cdot), \rho_i(\cdot))$, $i \in \mathbf{N}$, are weakly measurable. Clearly they are also weakly measurable and, in addition, compact valued, with respect to the weak topology of X. Thus, by virtue of [**5**, Theorem 4.1] Σ is measurable with respect to the weak topology of X. Since $S(x, r)$ is closed in the weak topology, the set $\{\omega \in \Omega | \Sigma(\omega) \cap S(x, r) \neq \varnothing\}$ belongs to $\mathscr{A}$ for every $x \in X$ and $0 < r < +\infty$. From this and the fact that every open (in the strong topology) set in X is a countable union of closed balls (for X separable) it follows that Σ is weakly measurable with respect to the strong topology of X.

THEOREM 1. *Let Ω be a measurable space. Let X and Y be separable Hilbert spaces and let D be a subset of X. Then every random contraction $f\colon \Omega \times D \to Y$ can be extended to a random contraction defined on all $\Omega \times X$.*

PROOF. First we show that f can be extended to a random contraction $\tilde{f}\colon \Omega \times (D \cup \{x_0\}) \to Y$, where $x_0 \in X \setminus D$. For this, let $\{a_i\}$ be a dense countable subset of D. For $\omega \in \Omega$, set

$$\Gamma(\omega) = \bigcap_{i=1}^{\infty} S(f(\omega, a_i), |a_i - x_0|). \tag{2}$$

We claim that for every $\omega \in \Omega$, $\Gamma(\omega) \neq \varnothing$. Otherwise we would have $\Gamma(\omega_0) = \varnothing$ for some $\omega_0 \in \Omega$. Since $|f(\omega_0, a_i) - f(\omega_0, a_j)| \leq |a_i - a_j|$ and $\bigcap_{i=1}^{\infty} S(a_i, |a_i - x_0|) \neq \varnothing$, property (K) furnishes the contradiction.

By Lemma 1 the multifunction Γ is weakly measurable and so, by [**6**], it admits a measurable selection, say γ. It is easy to see that a map $\tilde{f}\colon \Omega \times (D \cup \{x_0\}) \to Y$ given by $\tilde{f}(\omega, x) = f(\omega, x)$, if $(\omega, x) \in \Omega \times D$ and $\tilde{f}(\omega, x) = \gamma(\omega)$, if $(\omega, x) \in \Omega \times \{x_0\}$, is an admissible extension.

Now let A be a dense countable subset of $X \setminus D$. By induction f can be extended to a random contraction defined on $\Omega \times (D \cup A)$. Since $D \cup A$ is dense in X, by a standard argument f can be extended to a random contraction defined on $\Omega \times X$.

COROLLARY 1. *Let Ω, X, Y, D be as in Theorem 1, let $f\colon \Omega \times D \to Y$ be such that for every $x \in D$, $f(\cdot, x)$ is measurable and, for every $\omega \in \Omega$, $f(\omega, \cdot)$ is Lipschitz with constant $L(\omega)$. In addition suppose that L is a measurable function from Ω to $(0, +\infty)$. Then f can be extended to a map $\tilde{f}\colon \Omega \times X \to Y$ such that for every $x \in X$, $\tilde{f}(\cdot, x)$ is measurable and, for every $\omega \in \Omega$, $\tilde{f}(\omega, \cdot)$ is Lipschitz with the same constant $L(\omega)$.*

COROLLARY 2. *Let Ω be a measurable space and let Y be a separable Hilbert space. Let f_i, $g_i\colon \Omega \to Y$ ($i \in \mathbf{N}$) be measurable functions such that $|g_i(\omega) - g_j(\omega)| \leq |f_i(\omega) - f_j(\omega)|$ for every $\omega \in \Omega$ and every $i, j \in \mathbf{N}$. Let $f\colon \Omega \to Y$ be a measurable function. Then there exists a measurable function $g\colon \Omega \to Y$ such that $|g_i(\omega) - g(\omega)| \leq |f_i(\omega) - f(\omega)|$ for every $\omega \in \Omega$ and every $i \in \mathbf{N}$.*

PROOF. By property (K) and Lemma 1 we conclude that the multifunction $\Gamma\colon \Omega \to 2^Y$ given by $\Gamma(\omega) = \bigcap_{i=1}^{\infty} S(g_i(\omega), |f_i(\omega) - f(\omega)|)$ is well defined and weakly measurable. Thus it admits a measurable selection, say g. Clearly g satisfies the statement of Corollary 2.

THEOREM 2. *Let Ω be a complete measurable space. Let X and Y be separable metric spaces such that the pair (X, Y) has the property* (K). *Let D be a proper subset of X. Then every random contraction $f\colon \Omega \times D \to Y$ can be extended to a random contraction defined on all $\Omega \times X$.*

PROOF. Runs as that of Theorem 1. In the place of the multifunction given by (2) we consider the multifunction $\Gamma\colon \Omega \to 2^X$ given by

$$\Gamma(\omega) = \bigcap_{i=1}^{\infty} S(f(\omega, a_i), d_1(a_i, x_0)).$$

The measurability of Γ follows from [**5**, Theorem 3.5(iii)].

COROLLARY 3. *Let X and Y be normed spaces and let Ω and D be as in Theorem* 2. *Suppose that the pair (X, Y) has property* (K). *Let $f\colon \Omega \times D \to Y$ be such that for every $x \in D$, $f(\cdot, x)$ is measurable and, for every $\omega \in \Omega$, $f(\omega, \cdot)$ is Lipschitz with constant $L(\omega)$. In addition suppose that L is a measurable function from Ω to $\mathbf{R}$. Then f can be extended to a map $\tilde{f}\colon \Omega \times X \to Y$ such that for every $x \in X$, $\tilde{f}(\cdot, x)$ is measurable and, for every $\omega \in \Omega$, $\tilde{f}(\omega, \cdot)$ is Lipschitz with the same constant $L(\omega)$.*

Adopting the argument of the proof of [**8**, Theorem 11.3] and using the above results one can obtain the following

COROLLARY 4. *Let Ω be a complete measurable space. Let X be a separable Hilbert space and let D be a subset of X. Suppose that $f\colon \Omega \times D \to X$ is such that for every $x \in D$, $f(\cdot, x)$ is measurable and, for every $\omega \in \Omega$, $f(\omega, \cdot)$ is a Lipschitz-Hölder map of order α, $0 < \alpha \leq 1$ (i.e. $|f(\omega, x_1) - f(\omega, x_2)| \leq L(w)|x_1 - x_2|^{\alpha}$, $x_1, x_2 \in D$). In addition suppose that L is a measurable function from Ω to $(0, +\infty)$. Then f can be extended to a map $\tilde{f}\colon \Omega \times X \to Y$ preserving measurability with respect to the first variable and the Lipschitz-Hölder condition with respect to the second variable with the same constant $L(\omega)$.*

THEOREM 3. *Let Ω be a measurable space. Let X be a finite-dimensional Hilbert space and let D be a closed subset of X. Let $f\colon \Omega \times D \to X$ be a random isometry (i.e. for every $x \in D$, $f(\cdot, x)$ is measurable and, for every $\omega \in \Omega$, $f(\omega, \cdot)$ is an isometry). Then f can be extended to a random isometry defined on $\Omega \times X$.*

PROOF. Fix $u \in D$ and set $\widehat{D} = D - u$. Consider the map $g\colon \Omega \times \widehat{D} \to X$ given by $g(\omega, x) = f(\omega, x + u) - f(\omega, u)$. Clearly g is a random isometry. For every $\omega \in \Omega, g(\omega, 0) = 0$, and so, by [**7**], $g(\omega, \cdot)$ is linear. Observe that $|g(\omega, x)| = |x|$ and

$\langle g(\omega,x), g(\omega,y)\rangle = \langle x,y\rangle$, $\omega\in\Omega$, $x,y\in X$. (Here $|\cdot|$ stands for the norm and $\langle\cdot,\cdot\rangle$ for the inner product.) Let V be the linear span of $\widehat{D}$. Define $\hat{g}\colon \Omega\times V\to X$ by

$$\hat{g}\left(\omega, \sum_{i=1}^{n}\alpha_i x_i\right) = \sum_{i=1}^{n}\alpha_i g(\omega, x_i), \qquad x_1,\dots,x_n\in\widehat{D},\ \alpha_1,\dots,\alpha_n\in\mathbf{R}.$$

Observe that $\hat{g}$ is a random isometry.

Consider the multifunction $\Gamma\colon\Omega\to 2^X$ given by $\Gamma(\omega)=\hat{g}(\omega,V)$. By [**5**, Theorem 6.5] Γ is weakly measurable. Moreover, for every $\omega\in\Omega$, $\dim\Gamma(\omega)=\dim V$. Denote by $V^\perp$ and $\Gamma^\perp(\omega)$ the orthogonal complement of V and $\Gamma(\omega)$, respectively. Clearly $\dim V^\perp = \dim\Gamma^\perp(\omega)$. Let $\{\gamma_i\colon i\in\mathbf{N}\}$ be a measurable representation of Γ (see [**1**, Theorem III.6]). For $i\in\mathbf{N}$ consider the multifunction $\Gamma_i\colon\Omega\to 2^X$ given by $\Gamma_i(\omega)=\{x\in X|\langle\gamma_i(\omega),x\rangle=0\}$. Since for any $y\in X$, $\operatorname{dist}(y,\Gamma_i(\omega)) = |\langle\gamma_i(\omega),y\rangle|/|\gamma_i(\omega)|$, it follows from [**5**, Theorem 3.3] that Γ_i is weakly measurable. From the equality $\Gamma^\perp(\omega)=\bigcup_{n\ge1}(\bigcap_{i\ge1}\Gamma_i(\omega)\cap S(0,n))$ and [**5**, Theorem 4.1 and Proposition 2.3] it follows that $\Gamma^\perp$ is measurable.

Let ∂S denote the boundary of the unit ball S of X, and let $k=\dim V^\perp$. By [**5**, Proposition 2.4] the multifunction $\Gamma^\perp\cap\partial S$ is measurable. Let $\xi_1\colon\Omega\to X$ be a measurable selection of $\Gamma^\perp\cap\partial S$. Define $\Sigma_1\colon\Omega\to 2^X$ by $\Sigma_1(\omega)=\{x\in X|\langle x,\xi_1(\omega)\rangle=0\}$. The above argument shows that Σ_1 is measurable. By [**5**, Theorem 4.1] the multifunction $\Gamma^\perp\cap\Sigma_1\cap\partial S$ is measurable. Let ξ_2 be a measurable selection of $\Gamma^\perp\cap\Sigma_1\cap\partial S$. By recurrence one can construct functions $\xi_2,\dots,\xi_k$ such that ξ_i $(i=2,\dots,k)$ is a measurable selection of $\Gamma^\perp\cap\Sigma_1\cap\cdots\cap\Sigma_{i-1}\cap\partial S$. (Here $\Sigma_i(\omega)=\{x\in X|\langle x,\xi_i(\omega)\rangle=0\}$.) Note that for every fixed $\omega\in\Omega$, the vectors $\xi_1(\omega),\dots,\xi_k(\omega)$ are orthonormal.

Let $e_1,\dots,e_k$ be an orthonormal base $V^\perp$. Clearly the function $h\colon\Omega\times\{e_1,\dots,e_k\}\to X$ given by $h(\omega,e_i)=\xi_i(\omega)$ is a random isometry. As above, h can be extended to all $\Omega\times V^\perp$. Since for every $(\omega,x)\in\Omega\times V^\perp$, $h(\omega,x)\in\Gamma^\perp(\omega)$, there is a random linear isometry $\tilde{g}$ of all $\Omega\times X$ onto X that extends g (cf. [**4**, Theorem 3]). To finish the proof it suffices to observe that the random isometry $\tilde{f}\colon\Omega\times X\to Y$ given by $\tilde{f}(\omega,x)=\tilde{g}(\omega,x-u)+f(\omega,u)$ is an admissible extension.

REMARK. Theorem 3 fails, even in deterministic case, if X is supposed to be infinite-dimensional (see [**8**]).

The next theorem is a random version of Grunbaum's result [**2**] concerning the properties of monotone sets.

THEOREM 4. *Let Ω be a measurable space and let Y be a separable Hilbert space. Let $f_i, g_i\colon\Omega\to Y$ $(i=1,\dots,n)$ be measurable functions such that*

$$\langle f_i(\omega)-f_j(\omega), g_i(\omega)-g_j(\omega)\rangle\ge 0, \qquad \omega\in\Omega,\ i,j=1,\dots,n.$$

Let $\rho\colon\Omega\to(0,+\infty)$ be a measurable function.

Then there is a measurable function $h\colon\Omega\to Y$ such that

$$\langle f_i(\omega)+\rho(\omega)h(\omega), g_i(\omega)-h(\omega)\rangle\ge 0, \qquad \omega\in\Omega,\ i=1,\dots,n.$$

PROOF. From [**2**] it follows that for every fixed $\omega\in\Omega$, there is a vector $u\in Y$ (more precisely $u\in\operatorname{span}\{f_1(\omega),\dots,f_n(\omega),g_1(\omega),\dots,g_n(\omega)\}$) such that

$$(3)\qquad \langle f_i(\omega)+\rho(\omega)u, g_i(\omega)-u\rangle\ge 0, \qquad i=1,2,\dots,n.$$

A simple calculation shows that for every vector u satisfying (3) we have

$$|u| \le \sigma(\omega) \tag{4}$$

where $\sigma(\omega) = (\beta(\omega) + \sqrt{\beta^2(\omega) + 4\alpha(\omega)\rho(\omega)})/2\rho(\omega)$, $\alpha(\omega) = \sup\{|\langle f_i(\omega), g_i(\omega)\rangle|: i = 1, 2, \ldots, n\}$, $\beta(\omega) = \sup\{|\rho(\omega)g_i(\omega) - f_i(\omega)|: i = 1, 2, \ldots, n\}$. Note that a multifunction $\Sigma\colon \Omega \to 2^Y$ given by $\Sigma(\omega) = S(0, \sigma(\omega))$ is weakly measurable, for $\sigma(\cdot)$ being measurable.

For $i = 1, 2, \ldots, n$ and $k \in \mathbf{N}$ consider $\Gamma_i^k\colon \Omega \to 2^Y$ given by

$$\Gamma_i^k(\omega) = \mathrm{cl}\{y \in Y \,|\, \langle f_i(\omega) + \rho(\omega)y, g_i(\omega) - y\rangle > -1/k\}.$$

(Here $\mathrm{cl}\{\cdots\}$ denotes the closure of the set $\{\cdots\}$.) By virtue of [**2** and **5**, Theorem 6.2 and Proposition 2.6], Γ_i^k is nonempty valued and weakly measurable. Consider the multifunction $\Gamma\colon \Omega \to 2^Y$ given by

$$\Gamma(\omega) = \Sigma(\omega) \cap \bigcap_{\substack{i=1,\ldots,n \\ k \in \mathbf{N}}} \Gamma_i^k(\omega).$$

From [**2**] and (4) it follows that Γ is nonempty valued. Since Σ and Γ_i^k are weakly measurable with respect to the weak topology of Y and, in addition, Σ is weakly compact valued, by [**5**, Theorem 4.1] Γ is measurable with respect to the weak topology of Y, and so weakly measurable with respect to the strong topology of Y, Y being separable. Let h be a measurable selection of Γ. It is easy to see that h satisfies the statement of Theorem 4.

3. Acknowledgement. The authors wish to thank the referee for extremely attentive reading and useful critical remarks.

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