

THE SPACE OF CROSS SECTIONS OF A BUNDLE

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ABSTRACT. Let B be a nondiscrete compactum, Y a separable complete metrizable ANR with no isolated point and $p: X \rightarrow B$ a locally trivial bundle with fiber Y admitting a section. It is proved that the space $\Gamma(X)$ of all cross sections of $p: X \rightarrow B$ is an l_2 -manifold.

0. Introduction. Through the paper, spaces are separable metrizable and maps are continuous. Let $p: X \rightarrow B$ be a locally trivial bundle with fiber Y , that is, each point $b \in B$ has a neighborhood U and a homeomorphism $\varphi: U \times Y \rightarrow p^{-1}(U)$ such that $p\varphi = \pi_U$, the projection to U . A map $s: B \rightarrow X$ is called a *cross section* of $p: X \rightarrow B$ provided $ps = \text{id}$. The space of all cross sections of $p: X \rightarrow B$ with compact-open topology is denoted by $\Gamma(X)$. Then $\Gamma(X)$ is a closed subspace of the space $C(B, X)$ of all maps from B into X . If B is compact and d is a compatible metric for X , the topology of $\Gamma(X)$ (and $C(B, X)$) is induced by the sup-metric

$$\hat{d}(f, g) = \sup\{d(f(b), g(b)) \mid b \in B\}.$$

A manifold modeled on Hilbert space l_2 is called an l_2 -manifold. In this note, we prove the following

MAIN THEOREM. *Let B be a nondiscrete compactum, Y a complete metrizable ANR with no isolated point and $p: X \rightarrow B$ a locally trivial bundle with fiber Y admitting a section. Then $\Gamma(X)$ is an l_2 -manifold.*

For the trivial bundle $\pi_B: B \times Y \rightarrow B$, the space $\Gamma(B \times Y)$ can be regarded as the space $C(B, Y)$. Thus *the space $C(B, Y)$ is an l_2 -manifold if B is a nondiscrete compactum and Y is a complete-metrizable ANR with no isolated point.* This is a generalization of Eells-Geoghegan-Toruńczyk's result [E, Ge, To₁].

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1. Preliminaries. Our proof is based on the following:

TORUŃCZYK'S CHARACTERIZATION THEOREM FOR l_2 -MANIFOLDS [To₂] (CF. [To₃]). *A complete-metrizable ANR X is an l_2 -manifold if and only if X has the discrete approximation property, that is, for each map $f: \bigoplus_{n \in \mathbb{N}} I^n \rightarrow X$ of the free union of n -cells ($n \in \mathbb{N}$) into X and each map $\varepsilon: X \rightarrow (0, 1)$ there is a*

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map $g: \bigoplus_{n \in \mathbb{N}} I^n \rightarrow X$ with $d(f(x), g(x)) < \varepsilon f(x)$ for all $x \in \bigoplus_{n \in \mathbb{N}} I^n$ and $\{g(I^n) \mid n \in \mathbb{N}\}$ a discrete family in X .

In order to verify the discrete approximation property of $\Gamma(X)$, we use the following easy modification of [DT, Remark 2] (cf. Proof of [DT, Lemma 1]).

LEMMA 1. Let $X = (X, d)$ be a locally path-connected metric space with a tower $X_1 \subset X_2 \subset \dots (\subset X)$ satisfying the following properties:

(a) Given a compactum $A \subset I^n$, a map $f: I^n \rightarrow X$ with $f(A) \subset X_i$ and $\varepsilon > 0$, there is a map $g: I^n \rightarrow X_j$ into some $X_j \supset X_i$ with $f|_A = g|_A$ and $\hat{d}(f, g) < \varepsilon$;

(b) Given $\varepsilon > 0$, there is a $\delta > 0$ such that any map $f: I^n \rightarrow X_i$ is ε -homotopic to a map $g: I^n \rightarrow X_j$ in some $X_j \supset X_i$ with $\text{dist}(f(I^n), g(I^n)) > \delta$.

Then X has the discrete approximation property.

By the next lemma, we can treat noncompact ANR like a compact one.

LEMMA 2 [Mi, LEMMA 2.1]. Every metric ANR $Y = (Y, d_0)$ has a compatible metric $d \geq d_0$ with the following property:

(h) For each $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that any two δ -near maps of an arbitrary space to $Y = (Y, d)$ are ε -homotopic

Let $p: X \rightarrow B$ and $q: Z \rightarrow B$ be given maps. A map $f: Z \rightarrow X$ is said to be fiber-preserving (f.p.) if $pf = q$. We call $p: X \rightarrow B$ an absolute neighborhood fiber extensor (ANFE) if any f.p. map $f: A \rightarrow X$ of a closed set in an arbitrary space Z with a map $q: Z \rightarrow B$ can be extended to an f.p. map $\tilde{f}: U \rightarrow X$ of a neighborhood of A in Z . A map $p: X \rightarrow B$ is called an absolute neighborhood fiber retract (ANFR) provided whenever X is embedded in a space Z with a map $q: Z \rightarrow B$ as a closed set with $p = q|_X$ there is an f.p. retraction $r: U \rightarrow X$ of a neighborhood of X in Z onto X . A map $p: X \rightarrow B$ is an ANFR if and only if p is an ANFE. (See §1 of [Ya].) By [Ya, 1.2], we have the following

LEMMA 3. Any locally trivial bundle $p: X \rightarrow B$ with ANR fiber is an ANFR, that is, an ANFE.

We refer to [Du, Chapter XII] for function spaces and to [Hu] for ANR's.

2. Proof of main theorem. First we show that $\Gamma(X)$ is an ANR. Let $f: A \rightarrow \Gamma(X)$ be a map from a closed set in a space Z . Then f induces the map $F: A \times B \rightarrow X$ with $pF = \pi_B$, the projection. By Lemma 3, F extends to a map $\tilde{F}: W \rightarrow X$ from a neighborhood of $A \times B$ in $Z \times B$ with $p\tilde{F} = \pi_B$. From compactness of B , there is a neighborhood U of A in Z that $U \times B \subset W$. Then $\tilde{F}|_{U \times B}$ induces the map $\tilde{f}: U \rightarrow \Gamma(X)$ which is an extension of f . Hence $\Gamma(X)$ is an ANR.

Since Y is complete-metrizable, X is locally complete-metrizable, hence complete-metrizable (cf. [BP, Chapter II, Theorem 4.1]). Since B is nondiscrete, B has a cluster point b_∞ . From local triviality of p , there is an open neighborhood U of b_∞ in B and a homeomorphism $\varphi: \text{cl}U \times Y \rightarrow p^{-1}(\text{cl}U)$ such that $p\varphi = \pi_{\text{cl}U}$. Since Y has no isolated point, Y admits a compatible complete metric d_Y such that each component of Y has $\text{diam} > 1$ (cf. [BP, Chapter II, Theorem 3.2]). We may assume that (Y, d_Y) has the property (h) in Lemma 2. Let ρ be the product metric on $\text{cl}U \times Y$ defined by a metric for $\text{cl}U$ and d_Y . And let d be a compatible complete metric for X extending the metric on $p^{-1}(\text{cl}U)$ induced from ρ by φ (cf.

[BP, Chapter II, Theorem 3.2]). Then the sup-metric \hat{d} on $\Gamma(X)$ is a compatible complete metric.

Let $(b_i)_{i \in \mathbf{N}}$ be a sequence of distinct points in $U \setminus \{b_\infty\}$ which converges to b_∞ . For each $i \in \mathbf{N}$, let

$$\Gamma_i(X) = \{s \in \Gamma(X) \mid \pi_Y \varphi^{-1} s(b_j) = \pi_Y \varphi^{-1} s(b_\infty) \text{ for all } j \geq i\}.$$

Thus we have a tower $\Gamma_1(X) \subset \Gamma_2(X) \subset \dots (\subset \Gamma(X))$. We will show that this tower satisfies the properties (a) and (b) relative to \hat{d} . Then $\Gamma(X)$ is an l_2 -manifold by Toruńczyk's characterization.

(a) Let $A \subset I^n$ be a compactum, $f: I^n \rightarrow \Gamma(X)$ a map with $f(A) \subset \Gamma_i(X)$ and $\varepsilon > 0$. Then f induces the map $F: I^n \times B \rightarrow X$ such that $pF = \pi_B$ and

$$\pi_Y \varphi^{-1} F(a, b_k) = \pi_Y \varphi^{-1} F(a, b_\infty) \quad \text{for all } a \in A \text{ and } k \geq i.$$

Let $\delta = \delta(\varepsilon) > 0$ in the property (h) for (Y, d_Y) . Since $I^n \times B$ is compact, $\pi_Y \varphi^{-1} F$ is uniformly continuous. Thus we can choose $j \geq i$ so that

$$d_Y(\pi_Y \varphi^{-1} F(z, b_k), \pi_Y \varphi^{-1} F(z, b_\infty)) < \delta \quad \text{for all } z \in I^n \text{ and } k \geq j.$$

By using (h) and the Homotopy Extension Theorem [Hu, Chapter IV, Theorem 1.2], we can obtain a map $g': I^n \times \text{cl}U \rightarrow Y$ such that

$$\begin{aligned} g'|A \times \text{cl}U \cup I^n \times \text{bd}U &= \pi_Y \varphi^{-1} F|A \times \text{cl}U \cup I^n \times \text{bd}U, \\ g'(z, b_k) &= \pi_Y \varphi^{-1} F(z, b_\infty) \quad \text{for all } z \in I^n \text{ and } j \leq k \leq \infty, \\ g' \text{ and } \pi_Y \varphi^{-1} F|I^n \times \text{cl}U &\text{ are } \varepsilon\text{-homotopic.} \end{aligned}$$

Define a map $G: I^n \times B \rightarrow X$ with $pG = \pi_B$ as follows:

$$G(z, b) = \begin{cases} \varphi(b, g'(z, b)) & \text{if } b \in \text{cl}U, \\ F(z, b) & \text{otherwise.} \end{cases}$$

Then G induces the map $g: I^n \rightarrow \Gamma_j(X)$ such that $g|A = f|A$ and for each $z \in I^n$,

$$\begin{aligned} \hat{d}(g(z), f(z)) &= \sup\{d(g(z)(b), f(z)(b)) \mid b \in \text{cl}U\} \\ &= \sup\{d_Y(\pi_Y \varphi^{-1} G(z, b), \pi_Y \varphi^{-1} F(z, b)) \mid b \in \text{cl}U\} \\ &= \sup\{d_Y(g'(z, b), \pi_Y \varphi^{-1} F(z, b)) \mid b \in \text{cl}U\} < \varepsilon. \end{aligned}$$

(b) For each $\varepsilon > 0$, let $\delta = \delta(\varepsilon) > 0$ in (h) for (Y, d_Y) . We may assume $\delta < 1$. Let $f: I^n \rightarrow \Gamma_i(X)$ be a map. Then f induces the map $F: I^n \times B \rightarrow X$ such that $pF = \pi_B$ and

$$\pi_Y \varphi^{-1} F(z, b_k) = \pi_Y \varphi^{-1} F(z, b_\infty) \quad \text{for all } z \in I^n \text{ and } k \geq i.$$

From compactness, there exist $y_1, \dots, y_m \in Y$ such that

$$\pi_Y \varphi^{-1} F(I^n \times B) \subset B(y_1, \delta/3) \cup \dots \cup B(y_m, \delta/3),$$

where $B(y, r) = \{x \in Y \mid d_Y(x, y) < r\}$. For each $j = 1, \dots, m$, choose a point $z_j \in B(y_j, \delta) \setminus B(y_j, 2\delta/3)$. (Since each component of Y had $\text{diam} > 1$, we can choose such a point.) Using the Homotopy Extension Theorem and (h), we have a map $g_j: Y \rightarrow Y$ such that $g_j(B(y_j, \delta/3)) = z_j$, $g_j|Y \setminus B(y_j, \delta) = \text{id}$ and g_j is ε -homotopic to id . Then it follows

$$(*) \quad \max\{d_Y(y, g_j(y)) \mid j = 1, \dots, m\} > \delta/3 \text{ for each } y \in \pi_Y \varphi^{-1} F(I^n \times \text{cl}U).$$

Again using the Homotopy Extension Theorem, we can obtain a map $g': I^n \times \text{cl } U \rightarrow Y$ such that

$$\begin{aligned} g'|I^n \times \text{bd } U &= \pi_Y \varphi^{-1} F|I^n \times \text{bd } U, \\ g'(z, b_k) &= \pi_Y \varphi^{-1} F(z, b_\infty) \quad \text{if } i + m < k \leq \infty, \\ g'(z, b_{i+j}) &= g_j \pi_Y \varphi^{-1} F(z, b_{i+j}) \quad \text{if } j = 1, \dots, m, \\ g' &\text{ is } \varepsilon\text{-homotopic to } \pi_Y \varphi^{-1} F|I^n \times \text{cl } U \text{ rel. } I^n \times \text{bd } U. \end{aligned}$$

As in the proof of (a), define a map $G: I^n \times B \rightarrow X$ with $pG = \pi_B$ by using the above g' . It is easy to see that G is f.p. ε -homotopic to F . Then G induces the map $g: I^n \rightarrow \Gamma_{i+m+1}(X)$ which is ε -homotopic to f . We show $\text{dist}(f(I^n), g(I^n)) \geq \delta/6$. Suppose that $\hat{d}(f(z), g(z')) < \delta/6$ for some $z, z' \in I^n$. Then

$$\begin{aligned} d_Y(\pi_Y \varphi^{-1} F(z, b_\infty), \pi_Y \varphi^{-1} F(z', b_\infty)) &= d_Y(\pi_Y \varphi^{-1} F(z, b_\infty), g'(z', b_\infty)) \\ &= d(F(z, b_\infty), G(z', b_\infty)) \leq \hat{d}(f(z), g(z')) < \delta/6. \end{aligned}$$

And for each $j = 1, \dots, m$,

$$\begin{aligned} d_Y(\pi_Y \varphi^{-1} F(z, b_\infty), g_j \pi_Y \varphi^{-1} F(z', b_\infty)) \\ &= d_Y(\pi_Y \varphi^{-1} F(z, b_{i+j}), g_j \pi_Y \varphi^{-1} F(z', b_{i+j})) \\ &= d_Y(\pi_Y \varphi^{-1} F(z, b_{i+j}), g'(z', b_{i+j})) = d(F(z, b_{i+j}), G(z', b_{i+j})) \\ &\leq \hat{d}(f(z), g(z')) < \delta/6. \end{aligned}$$

Hence for each $j = 1, \dots, m$,

$$d_Y(\pi_Y \varphi^{-1} F(z', b_\infty), g_j \pi_Y \varphi^{-1} F(z', b_\infty)) < \delta/3.$$

This contradicts (*). Therefore $d(f(z), g(z')) \geq \delta/6$ for any $z, z' \in I^n$, that is, $\text{dist}(f(I^n), g(I^n)) \geq \delta/6$. The proof is completed. \square

3. The space of fiber-preserving maps. Let $p: X \rightarrow B$ and $q: Z \rightarrow B$ be given maps. By $C_B(Z, X)$, we denote the space of f.p. maps from Z to X with compact-open topology. The following is a generalization of Main Theorem which can be proved directly by the same method but it follows from Main Theorem as a corollary.

COROLLARY. *Let $q: Z \rightarrow B$ be a map of a nondiscrete compactum Z , Y a complete-metrizable ANR with no isolated point and $p: X \rightarrow B$ a locally trivial bundle with fiber Y . Then $C_B(Z, X)$ is an l_2 -manifold if $C_B(Z, X) \neq \emptyset$.*

PROOF. Consider the fiber-product $X \times_B Z = \{(x, z) \in X \times Z \mid p(x) = q(z)\}$ of X and Z over B . Let $\pi_X: X \times Z \rightarrow X$ and $\pi_Z: X \times Z \rightarrow Z$ denote the projections. Then $\pi_Z|X \times_B Z$ is a locally trivial bundle with fiber Y because so is $p: X \rightarrow B$. Let $\Gamma(X \times_B Z)$ be the space of cross-sections of this bundle. Define a map $\theta: \Gamma(X \times_B Z) \rightarrow C_B(Z, X)$ by $\theta(s) = \pi_X \circ s$. Then it is easy to see that θ is a homeomorphism. The result follows from Main Theorem. \square

As mentioned in Lemma 3, any locally trivial bundle with ANR fiber is an ANFR. Besides a Hurewicz fibration between ANR's and a proper strongly regular map onto a finite-dimensional space with ANR fibers are also ANFR's (cf. [Ya, 1.2]). We have the f.p. Homotopy Extension Theorem [Ya, 1.1 (iv)] and the f.p. version of Lemma 2. By the same method as Main Theorem, we can prove the following theorem which is analogous to the above corollary but in a little different setting.

THEOREM. *Let $q: Z \rightarrow B$ be a map of a compactum and $p: X \rightarrow B$ an ANFR such that X is complete-metrizable. Then $C_B(Z, X)$ is an l_2 -manifold if $C_B(Z, X) \neq \emptyset$, $q^{-1}(b)$ is nondiscrete and $p^{-1}(b)$ has no isolated point for some $b \in B$.*

Because of similarity, the proof is left to the reader. (We can also prove the relative version of this theorem, i.e., the f.p. version of [To₁, Theorem 5.5].) Here we give the proof of the f.p. version of Lemma 2.

LEMMA 2'. *Let $p: X \rightarrow B$ be an ANFR and d_0 a metric for X . Then X has a compatible metric $d \geq d_0$ with the following property*

(h') *For each $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that any two δ -near f.p. maps of an arbitrary space Z with a map $q: Z \rightarrow B$ to $X = (X, d)$ are f.p. ε -homotopic.*

PROOF. Since $X = (X, d_0)$ can be isometrically embedded in a normed linear space E as a closed set (cf. [BP, Chapter II, Corollary, 1.1]), we have an f.p. closed embedding $i: X \rightarrow B \times E$ such that $\pi_E i$ is an isometry, where $\pi_E: B \times E \rightarrow E$ is the projection. Identify X with $i(X) \subset B \times E$ and $p = \pi_E|_X$. Let d_1 be the product metric on $B \times E$. Then $d_1 \geq d_0$ on X . Since $p: X \rightarrow B$ is an ANFR, there is an f.p. retraction $r: G \rightarrow X$ of a neighborhood G of X in $B \times E$. For a subset $S \subset B \times E$, we denote

$$\text{conv}_B S = \bigcup \{ \{b\} \times \text{conv } \pi_E(S \cap \{b\} \times E) \mid b \in B \},$$

where $\text{conv } A$ denotes the convex hull of $A \subset E$. For each $x \in X$ and each neighborhood V of x in X , there is a basic open set $U = U_1 \times U_2$ in $B \times E$ such that $\pi_E(U) = U_2$ is convex and $x \in U \subset r^{-1}(V)$. Let $W = U \cap X$. Then $\text{conv}_B W \subset U \subset G$ and $r(\text{conv}_B W) \subset r(U) \subset V$. By the same way as [Mi, Lemma 2.1], we have a compatible metric $d \geq d_1$ on X with the following property:

(c) *To every $\varepsilon > 0$ corresponds a $\delta = \delta(\varepsilon) > 0$ such that if $S \subset X$ with $\text{diam } S < \delta$ then $\text{conv}_B S \subset G$ and $\text{diam } r(\text{conv}_B S) < \varepsilon$.*

By standard arguments, (h') follows from (c). (Cf. [Hu, Chapter IV, Theorem 1.1].) \square

REFERENCES

- [BP] C. Bassaga and A. Pełczyński, *Selected topics in infinite-dimensional topology*, Monogr. Mat. 58, PWN, Warsaw, 1975.
- [DT] T. Dobrowolski and H. Toruńczyk, *Separable complete ANR's admitting a group structure are Hilbert manifolds*, *Topology Appl.* **12** (1981), 229–235.
- [Du] J. Dugundji, *Topology*, Allyn & Bacon, Boston, Mass., 1966.
- [E] J. Eells, Jr., *On geometry of function spaces*, *Symp. Intern. de Topologia Algebraica Univ. Nacional Autónoma de México and UNESCO, Mexico City, 1958*, pp. 303–308.
- [Ge] R. Geoghegan, *On spaces of homeomorphisms, embeddings and functions, I*, *Topology* **11** (1972), 159–177.
- [Hu] S.-T. Hu, *Theory of retracts*, Wayne State Univ. Press, 1965.
- [Mi] E. Michael, *Uniform AR's and ANR's*, *Compositio Math.* **39** (1979), 129–139.
- [To₁] H. Toruńczyk, *Concerning locally homotopy negligible sets and characterization of l_2 -manifolds*, *Fund. Math.* **101** (1978), 93–110.
- [To₂] ———, *Characterizing Hilbert space topology*, *Fund. Math.* **111** (1981), 247–262.
- [To₃] ———, *A correction of two papers concerning Hilbert manifolds*, *Fund. Math.* **125** (1985), 89–93.
- [Ya] T. Yagasaki, *Fiber shape theory*, *Tsukuba J. Math.* **9** (1985), 261–277.

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