PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 103, Number 3, July 1988

CONTINUOUS IMAGES OF ARCS

JACEK NIKIEL

(Communicated by Dennis K. Burke)

ABSTRACT. A characterization of continuous images of (Hausdorff) arcs is used to describe their cyclic elements as inverse limits of inverse systems of 'nice' spaces with 'nice' bonding mappings.

Introduction. By a *continuum* we will mean a compact and connected Hausdorff space, and an *arc* will be understood as a Hausdorff arc, i.e., a continuum with exactly two non-cut-points. Recall that arcs are precisely compact and connected linearly ordered topological spaces, and each separable arc is homeomorphic to $[0,1]$ (see for example [**6**]).

The classical Hahn-Mazurkiewicz theorem [**5**, **10**] characterizes continuous images of $[0,1]$ (in the class of all Hausdorff spaces) as locally connected and metrizable continua. Since 1960 it is known that there exist locally connected continua which are continuous images of no arc ([**7**]; recall that each Hausdorff space which is a continuous image of some arc is a locally connected continuum—see for example [**6**]). Hence there is no straightforward generalization of the Hahn-Mazurkiewicz theorem. The first reasonable characterization of continuous images of arcs was found in 1985 in [**12**], where several natural equivalent conditions were proved. Some of those conditions involve the use of the cyclic element theory (see for example [**20**] for the cyclic element theory in metrizable locally connected continua, and [**21**, **2**] for generalizations of the theory to the class of all locally connected continua). The utility of the cyclic element theory as a way to find a nonseparable version of the Hahn-Mazurkiewicz theorem was noticed for the first time in [**2**], where it was proved that if each cyclic element of a locally connected continuum X is a continuous image of some arc, then also the whole X is an image of some arc.

In the present paper we apply the characterization theorem of [**12**] to prove that if X is a cyclic element of a continuum which is a continuous image of some arc, then X is homeomorphic to the inverse limit of an inverse sequence of some 'piecewise' metrizable continua (each of which is also a continuous image of some arc and, moreover, can be defined by a simple induction) with natural bonding mappings. Moreover, we describe some examples and applications of the obtained results. Some further applications of the related methods can be found in [**14**].

2. Auxiliary facts. Let X be a locally connected continuum. We say that a subset A of X is a *T-set* in X provided A is a closed set and each component

Received by the editors November 21, 1986 and, in revised form May 7, 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 54B25, 54F25; Secondary 54C10.

Key words and phrases. Arc, cyclic element, Hahn-Mazurkiewicz theorem, inverse limit, monotone mapping.

0002-9939/88 $1.00 + $.25 per page

of $X - A$ has a two-point boundary. If X has no cut points, then we say that a sequence $A_1, A_2, \dots$ of T-subsets of X *T-approximates* X provided

(1) $A_1 \subset A_2 \subset \cdots$,

(2) if P is a component of $X - A_n$, $n = 1, 2, \dots$, then the set of all cut points of $\overline{P}$ is contained in A_{n+1},

(3) if P is a component of $X - A_n$, $n = 1, 2, \dots$, and C is a nondegenerate cyclic element of $\overline{P}$, then the set $C \cap A_{n+1}$ is metrizable and contains at least three points, and

(4) A_1 is metrizable.

It turns out that if X has no cut points and a sequence $A_1, A_2, \dots$ of T-subsets of X fulfills (1), (2) and

(3′) if P is a component of $X - A_n$, $n = 1, 2, \dots$, and C is a nondegenerate cyclic element of $\overline{P}$, then the set $C \cap A_{n+1}$ contains at least three points, then $A = \bigcup_{n=1}^{\infty} A_n$ is a dense subset of X (see [**12**, Lemma 3.4]).

Theorem 2.1. is a particular case of [**12**, Theorem 1.1]:

2.1. THEOREM. *If X is a continuum, then the following conditions are equivalent*:

(i) *X is a continuous image of some arc,*

(ii) *X is locally connected and if Y is a nondegenerate cyclic element of X and $p, q, r \in Y$ are any points, then there is a metrizable T-set A in Y such that $p, q, r \in A$,*

(iii) *X is locally connected and each nondegenerate cyclic element Y of X can be T-approximated by some sequence $A_1^Y, A_2^Y, \dots$ of T-subsets of Y.*

Let X and Y be locally connected continua, $f \colon X \to Y$ be a function and A be a T-set in X. We say that f is a *T-map* with respect to A provided

(5) f is a continuous surjection,

(6) $B = f(A)$ is a T-set in Y and $f\,|_A$ is a homeomorphism from A onto B,

(7) each component Q of $Y - B$ is homeomorphic to $]0, 1[$ (so $\overline{Q}$ is homeomorphic to $[0, 1]$), and

(8) for each component Q of $Y - B$ there is the unique component P_Q of $X - A$ such that $f(P_Q) \subset \overline{Q}$, and each component of $X - A$ is a P_Q for some component Q of $Y - B$.

Furthermore, $f \colon X \to Y$ is said to be a T-map if it is a T-map with respect to some T-subset A of X.

2.2. LEMMA. *If X is a locally connected continuum and A is a T-set in X, then there are a locally connected continuum X_A and a function $f_A \colon X \to X_A$ such that f_A is a T-map with respect to A. Moreover, X_A is determined uniquely up to a homeomorphism.*

PROOF. The first part of Lemma 2.2 was shown by L. B. Treybig, [**16**, Theorem 6], for the second part see [**12**, Lemma 3.5].

2.3. LEMMA. *If X is a locally connected continuum, A, B are T-sets in X such that $B \subset A$, and $f \colon X \to X_A$ is a T-map with respect to A, then $f(B)$ is a T-set in X_A. Moreover, if P is a component of $X - B$, then there is the unique component Q of $X_A - f(B)$ such that $Q \subset f(P) \subset \overline{Q}$.*

PROOF. Let P be a component of $X - B$, $\mathrm{bd}(P) = \{b_1, b_2\}$, and put $Q = f(P) - f(\{b_1, b_2\})$. It suffices to show that Q is a component of $X_A - f(B)$ such that $\mathrm{bd}(Q) = f(\{b_1, b_2\})$. Note that

$$f^{-1}f(b_i) \subset \{b_i\} \cup \bigcup\{U \colon U \text{ is a component of } X - A \text{ and } b_i \in \mathrm{bd}(U)\}.$$

The set $G = (X - P) \cup f^{-1}f(\{b_1, b_2\})$ is closed in X. Moreover, $Q = X_A - f(G)$ (because f is a T-map and $B \subset A$). Hence Q is an open subset of X_A. Observe that $\mathrm{bd}(Q) = f(\{b_1, b_2\})$. Suppose that Q is not connected and let V_1, V_2 be nonempty, disjoint and open subsets of Q such that $Q = V_1 \cup V_2$. Put

$$W_i = (f^{-1}(V_i) \cap A) \cup \bigcup\{U \colon U \text{ is a component of } X - A \text{ and } f(U) \cap V_i \neq \emptyset\},$$

for $i = 1, 2$. Note that W_1, W_2 are nonempty, disjoint and open subsets of P such that $P = W_1 \cup W_2$. Hence P is not connected, a contradiction.

2.4. LEMMA. *If X is a continuum without cut points which is a continuous image of some arc, $A_1, A_2, \ldots$ is a sequence of T-subsets of X which T-approximates X, and $f_n \colon X \to X_{A_n}$ is a T-map with respect to A_n, for $n = 1, 2, \ldots$, then the family $\{f_1, f_2, \ldots\}$ separates points of X.*

PROOF. Put $A = \bigcup_{n=1}^{\infty} A_n$; so A is dense in X. Let x and y be distinct points of X. We consider three cases.

Case 1. $x, y \in A$. Hence there is an integer n such that $x, y \in A_n$. Then $f_n(x) \neq f_n(y)$.

Case 2. $x \in A$ and $y \notin A$. By [**16**, Theorem 8], there is an integer n such that $x \in A_n$ and $x \notin \mathrm{bd}(P)$, where P is a component of $X - A_n$ such that $y \in P$. Therefore $f_n(x) \neq f_n(y)$.

Case 3. $x, y \notin A$. By [**16**, Theorem 8] there is an integer n such that if P, Q are components of $X - A_n$ so that $x \in P$, $y \in Q$, then $\mathrm{bd}(P) \cap \mathrm{bd}(Q) = \emptyset$. Hence $f_n(x) \neq f_n(y)$.

3. The main result. Let X be a continuum which is a continuous image of some arc (so X is locally connected). We say that the *rank* of X is less than or equal to a nonnegative integer m, $r(X) \leq m$, provided either

(9) each cyclic element of X is degenerate or

(10) X has a nondegenerate cyclic element and for each nondegenerate cyclic element Y of X there is a sequence $A_1^Y, A_2^Y, \ldots$ of T-subsets of Y which T-approximates Y and is such that $A_m^Y = Y$.

We say that the rank of X is equal to m provided $r(X) \leq m$ and either $m = 0$ or $r(X) \nleq m - 1$.

3.1. PROPOSITION. *If X is a continuum which is a continuous image of some arc, then*

(i) $r(X) = 0$ *if and only if X is a dendron,*

(ii) $r(X) \leq 1$ *if and only if each cyclic element of X is metrizable.*

3.2. THEOREM. *Let X be a continuum without cut points which is a continuous image of some arc. If $A_1, A_2, \ldots$ is a sequence of T-subsets of X which T-approximates X, then*

(i) $r(X_{A_n}) \leq n$ *for* $n = 1, 2, \ldots,$ *and*

(ii) *there are functions* $f_n\colon X \to X_{A_n}$ *and* $g_n\colon X_{A_{n+1}} \to X_{A_n}$, $n = 1, 2, \ldots$, *such that*

(a) *each* f_n *is a* T*-map with respect to* A_n,

(b) *each* g_n *is a* T*-map with respect to* $f_{n+1}(A_n)$, *and*

(c) *the diagrams*

$$\begin{array}{ccc} X & \xleftarrow{\ \mathrm{id}_X\ } & X \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f_{n+1}} \\ X_{A_n} & \xleftarrow[\ g_n\ ]{} & X_{A_{n+1}} \end{array}$$

commute, and so X *is homeomorphic to* $\lim \operatorname{inv}(X_{A_{n+1}}, g_n)$.

PROOF. For simplicity, we will write X_n instead of X_{A_n}.

(i) By Lemma 2.3, $S = \{f_n(A_1), f_n(A_2), \ldots, f_n(A_{n-1}), X_n\}$ is a sequence of T-subsets of X_n. To see that S T-approximates X_n it suffices to prove that if P is a component of $X_n - f_n(A_{n-1})$ and C is a nondegenerate cyclic element of $\overline{P}$, then the set $C \cap X_n = C$ is metrizable. Let Q denote the unique component of $X - A_{n-1}$ such that $f_n(Q) \subset \overline{P}$. There is the unique cyclic element D of $\overline{Q}$ such that $f_n(D) = C$. Note that $f_n \mid_D\colon D \to C$ is a T-map with respect to $A_n \cap D$ and $A_n \cap D$ is metrizable. By [**12**, Lemma 4.2], C is metrizable.

(ii) Note that, by [**12**, Lemma 4.2], X_1 is a metrizable space. Let $h_1\colon X \to X_1$ be any T-map with respect to A_1. Suppose that, for some positive integer n, the function $h_n\colon X \to X_1$ is already constructed such that h_n is a T-map with respect to A_1, $h_n\mid_{A_1} = h_1\mid_{A_1}$, and for each component P of $X - A_n$, $\operatorname{bd}(P) = \{p_1, p_2\}$, the set $h_n(\overline{P})$ is an arc (maybe degenerate) with endpoints $h_n(p_1)$, $h_n(p_2)$.

Let x be any point of X. If $x \in A_{n+1}$, then define $h_{n+1}(x) = h_n(x)$. Suppose that $x \notin A_{n+1}$ and let P denote a component of $X - A_{n+1}$ such that $x \in P$, $\operatorname{bd}(P) = \{p_1, p_2\}$. Let Q be a component of $X - A_1$ such that $P \subset Q$, $\operatorname{bd}(Q) = \{q_1, q_2\}$. Recall that $I = h_n(\overline{Q})$ is an arc with endpoints $h_n(q_1), h_n(q_2)$. We may assume that $h_n(q_1) \le h_n(p_1) \le h_n(p_2) \le h_n(q_2)$ in the fixed natural ordering of I. Define

$$h_{n+1}(x) = \begin{cases} h_n(x) & \text{if } h_n(p_1) \le h_n(x) \le h_n(p_2), \\ h_n(p_1) & \text{if } h_n(x) \le h_n(p_1), \\ h_n(p_2) & \text{if } h_n(p_2) \le h_n(x). \end{cases}$$

It is obvious that h_{n+1} is a T-map with respect to A_1 (the continuity of h_{n+1} follows from Lemma 3.1 of [**12**]).

Note that the sequence $h_1, h_2, \ldots$ uniformly converges to a function $f_1\colon X \to X_1$ (in the fixed metric on X_1; use for example the second part of Lemma 3.4 of [**12**]). Moreover, f_1 is a T-map with respect to A_1, and

(∗1) for each positive integer m and each component P of $X - A_m$, $\operatorname{bd}(P) = \{p_1, p_2\}$, the set $f_1(\overline{P})$ is an arc (maybe degenerate) from $f_1(p_1)$ to $f_1(p_2)$.

Suppose that for some positive integer n the function $f_n\colon X \to X_n$ is already constructed such that f_n is a T-map with respect to A_n, and

(∗n) for each positive integer m, $m \ge n$, and each component P of $X - A_m, \operatorname{bd}(P) = \{p_1, p_2\}$, the set $f_n(\overline{P})$ is an arc (maybe degenerate) from $f_n(p_1)$ to $f_n(p_2)$.

We define functions $f_{n+1}\colon X \to X_{n+1}$ and $g_n\colon X_{n+1} \to X_n$ such that f_{n+1} is a T-map with respect to A_{n+1} so that $(*n+1)$ holds, g_n is a T-map with respect to $f_{n+1}(A_n)$, and $f_n = g_n \circ f_{n+1}$.

Let $i\colon X \to X_{n+1}$ be any T-map with respect to A_{n+1}. If $x \in A_{n+1}$ then define $f_{n+1}(x) = i(x)$ and if $y \in i(A_{n+1})$, $y = i(x)$ for some $x \in A_{n+1}$, then set $g_n(y) = f_n(x)$. Let x be any point of $X - A_{n+1}$ and let P denote a component of $X - A_{n+1}$ such that $x \in P$, $\mathrm{bd}(P) = \{p, p'\}$. Let also Q be a component of $X_{n+1} - i(A_{n+1})$ such that $i(P) \subset \overline{Q}$. Recall that $\mathrm{bd}(Q) = \{i(p), i(p')\}$ and $\overline{Q}$ is homeomorphic to $[0,1]$. If $f_n(P)$ consists of a single point z, then put $f_{n+1}(x) = i(x)$ and $g_n(i(x)) = z$. Assume that $f_n(P)$ is nondegenerate. By $(*n)$, $f_n(\overline{P})$ is a metrizable arc with endpoints $f_n(p), f_n(p')$. Let $j\colon \overline{Q} \to f_n(\overline{P})$ be any homeomorphism such that $j(i(p)) = f_n(p)$ and put $f_{n+1}(x) = j^{-1}(z)$ provided $z = f_n(x)$. Set also $g_n(f_{n+1}(x)) = j(f_{n+1}(x))$.

Now, observe that $f_{n+1}\colon X \to X_{n+1}$ and $g_n\colon X_{n+1} \to X_n$ are well-defined functions. By Lemma 3.1 of [**12**], these functions are continuous. An easy straightforward proof shows that f_{n+1} and g_n fulfill all the required conditions.

Since $f_n = g_n \circ f_{n+1}$, for $n = 1, 2, \ldots$, there is an induced map $f\colon X \to \lim\mathrm{inv}(X_{n+1}, g_n)$. By Lemma 2.4, f is one-to-one. Since all the maps f_n are onto, also f is a surjection. Thus f is a homeomorphism.

4. Examples and remarks.

4.1. EXAMPLE. Let Z denote a θ-curve (i.e., Z is a union of a circle and its diameter) and let c, d denote the ramification points of Z. Let also a and b be any points of $Z - \{c, d\}$ which lie in distinct components of $Z - \{c, d\}$. Put $A' = \{a, b, c, d\}$.

Let $Y = \{0, 1\} \cup Z \times]0, 1[$. Let y be any point of Y. We define a family B_y of basic neighbourhoods of y in Y. If $y \in (Z - \{a, b\}) \times \{r\}$, for some $r \in]0, 1[$, then let B_y be the family of all $V \times \{r\}$, where V is an open subset of Z such that $y \in V \times \{r\}$ and $a, b \notin V$. If $y = 0$ then put

$$B_y = \{\{y\} \cup Z \times]0, r[\colon r \in]0, 1[\},$$

and if $y = 1$ then set

$$B_y = \{\{y\} \cup Z \times]r, 1[\colon r \in]0, 1[\}.$$

If $y = (a, r)$ for some $r \in]0, 1[$ then let

$$B_y = \{Z \times]t, r[\cup V \times \{r\} \colon t \in]0, r[, V \text{ is an open subset of } Z,\ a \in V \text{ and } b \notin V\},$$

and if $y = (b, r)$ for some $r \in]0, 1[$ then let

$$B_y = \{Z \times]r, t[\cup V \times \{r\} \colon t \in]r, 1[, V \text{ is an open subset of } Z,\ b \in V \text{ and } a \notin V\}.$$

Put also $A = \{0, 1\} \cup A' \times]0, 1[$. It is obvious that Y is a locally connected continuum which is a cyclic chain from 0 to 1, each nondegenerate cyclic element of Y is homeomorphic to Z, A is a T-set in Y which contains all cut points of Y, and each component of $Y - A$ is homeomorphic to $]0, 1[$. Define $f\colon Y \to [0, 1]$ by the formulas $f(0) = 0$, $f(1) = 1$, and $f(z, r) = r$ for $z \in Z$ and $r \in]0, 1[$. Then f is a continuous and monotone surjection.

Now, put $X_1 = Z$ and $A_1^1 = A'$. Suppose that a locally connected continuum X_n and a T-subset A_n^n of X_n are already constructed such that X_n has no cut points and each component of $X_n - A_n^n$ is homeomorphic to $]0, 1[$.

Put $S_n = \{P\colon P$ is a component of $X_n - A_n^n\}$. If $P \in S_n$ then denote $\mathrm{bd}(P) = \{e_0^P, e_1^P\}$ and let $f_P\colon \overline{P} \to [0,1]$ be any homeomorphism such that $f_P(e_0^P) = 0$. Put $X'_{n+1} = A_n^n \cup Y \times S_n$ and let ρ be an equivalence relation on X'_{n+1} the only nondegenerate classes of which are of the form

$$E_i^P = \{e_i^P\} \cup \{(i,Q) \in Y \times \{Q\}\colon Q \in S_n \text{ and } e_i^P \in \mathrm{bd}(Q)\}$$

for each $i \in \{0,1\}$ and $P \in S_n$ (note that the sets E_i^P are finite).

Let X_{n+1} denote the quotient set and $g\colon X'_{n+1} \to X_{n+1}$ be the quotient function. Let x be any point of X_{n+1}. We define a family C_x of basic neighbourhoods of x in X_{n+1}. If $x = g(y,P)$ for some $y \in Y - \{0,1\}$ and $P \in S_n$, then put $C_x = \{g(V \times \{P\})\colon V$ is open in Y, $y \in V$ and $0,1 \notin V\}$. If $x = g(x')$ for some $x' \in A_n^n$ then set

$$C_x = \left\{ g(U \cap A_n^n) \cup \bigcup_{P \in S_n} g(f^{-1} f_P(U \cap P) \times \{P\})\colon \right.$$
$$\left. U \text{ is an open neighbourhood of } x' \text{ in } X_n \right\}.$$

Put

$$A_n^{n+1} = g(A_n^n), \quad A_{n+1}^{n+1} = A_n^{n+1} \cup \bigcup_{P \in S_n} g(A \times \{P\})$$

and let

$$f_n(x) = \begin{cases} g^{-1}(x) \cap A_n^n & \text{if } x \in A_n^{n+1}, \\ f_P^{-1} f(y) & \text{if } x = g(y,P) \text{ for some } y \in Y \text{ and } P \in S_n. \end{cases}$$

An easy proof shows that X_{n+1} is a locally connected continuum without cut points, A_n^{n+1} and A_{n+1}^{n+1} are T-sets in X_{n+1}, and $f_n\colon X_{n+1} \to X_n$ is a T-map with respect to A_n^{n+1}. Moreover, f_n is monotone.

Now, we inductively define T-subsets A_n^m of X_m for any positive integers m, n so that $m \geq n$: suppose that A_n^m is already defined and put $A_n^{m+1} = A_m^{m+1} \cap f_m^{-1}(A_n^m)$.

Put $X = \lim\mathrm{inv}(X_{n+1}, f_n)$. Then X is a locally connected continuum (see [**1**, Theorem 4.3, p. 241]) which has no cut points. For each positive integer n, set $A_n = \lim\mathrm{inv}(A_n^{m+1}, f_m \mid_{A_n^{m+1}})$. Note that $A_1, A_2, \ldots$ is a sequence of T-subsets of X which T-approximates X. Thus X is a continuous image of some arc. It is not difficult to see that $r(X) \neq k$ for each nonnegative integer k.

Observe also that each point of X has arbitrarily small open neighborhoods the boundary of which consists of either two or three points. Thus X is a rim-finite continuum. This gives another proof that X is a continuous image of some arc (see [**15** and **18**]). Moreover, note that X fulfills the first axiom of countability.

4.2. REMARK. Let X be a continuum without cut points which is a continuous image of some arc and let $A_1, A_2, \ldots$ be a sequence of T-subsets of X which T-approximates X. One can use maps f_n, g_n of Theorem 3.2 to find a directed by inclusion family D of T-subsets of X and maps $i_A\colon X \to X_A$, $j_B^A\colon X_A \to X_B$, for $A, B \in D$, $B \subset A$, such that

(a) each i_A is a T-map with respect to A and each j_B^A is a T-map with respect to $i_A(B)$,

(b) each space X_A is metrizable,

(c) $T = (A, j_A^B, D)$ is an inverse system,
(d) $j_B^A \circ i_A = i_B$ for any $A, B \in D$, $B \subset A$, and
(e) X is homeomorphic to $\lim\operatorname{inv} T$.
(Compare this with the proof given in [**13**]).

Recall that each locally connected continuum is an inverse limit of some inverse system of metrizable locally connected continua with monotone bonding mappings [**8**, Theorem 2].

4.3. EXAMPLE. Let L denote the well-known long interval (see for example [**3**, p. 297]) and for each ordinal number α, $0 < \alpha < \omega_1$, let $f_\alpha\colon [0,1]\times L \to [0,1]\times[0,\alpha]_L$ be a function defined by the formula

$$f_\alpha(s,t) = \begin{cases} (s,t) & \text{if } t \leq_L \alpha, \\ (s,\alpha) & \text{if } \alpha \leq_L t, \end{cases}$$

where $\leq_L$ is the natural ordering of L from 0 to ω_1. Moreover, if α, β are ordinals, $0 < \alpha \leq \beta < \omega_1$, then put $f_\alpha^\beta = f_\alpha \mid_{[0,1]\times[0,\beta]_L}$. Note that f_α, f_α^β are monotone surjections and $f_\alpha^\beta \circ f_\beta^\gamma = f_\alpha^\gamma$ if $0 < \alpha \leq \beta \leq \gamma < \omega_1$. Therefore $[0,1] \times L$ is homeomorphic to lim inv $([0,1] \times [0,\beta]_L, f_\alpha^\beta, \omega_1)$. Moreover, for each $0 < \alpha < \omega_1$, $[0,1] \times [0,\alpha]_L$ is homeomorphic to the square $[0,1] \times [0,1]$, and so it is a continuous image of $[0,1]$. However, even in such a simple case as considered here, the inverse limit space $[0,1] \times L$ is not a continuous image of any arc or even any compact linearly ordered topological space ([**22**], see also [**17**]).

4.4. PROBLEM. Suppose that $T = (X_{n+1}, f_n)$ is an inverse sequence such that all the spaces X_n are continuous images of some arcs and all the mappings $f_n\colon X_{n+1} \to X_n$ are monotone surjections. Does it follow that the inverse limit space $X = \lim\operatorname{inv} T$ is a continuous image of any arc?

Recall that X is locally connected ([**1**], see also [**4**]).

4.5. REMARK. In [**12**] it was also proved that if a continuum X is a continuous image of some arc, then X can be approximated by some family J of finite dendrons (see [**19**, **17** or **12**] for the definition of the last notion). The proof given there can be simplified with the use of some ideas of the present paper. Namely, assume that X is a continuum without cut points which is a continuous image of some arc and let $A_1, A_2, \ldots$ be a sequence of T-subsets of X which T-approximates X. For each n let $f_n\colon X \to X_{A_n}$ and $g_n\colon X_{A_{n+1}} \to X_{A_n}$ be maps constructed in Theorem 3.2. In [**12**] it was shown that, for each n, there is a family J_n of finite dendrons in X_{A_n} such that

(a) J_n approximates X_{A_n} and $X_{A_n} = \bigcup J_n$ (so for every points $x, y \in X_{A_n}$ we may define $[x,y]_n$ to be the unique subarc of X_{A_n} such that x, y are the endpoints of $[x,y]_n$ and $[x,y]_n \subset T$ for some $T \in J_n$), and

(b) there is a point $a \in A_1$ such that $g_n([f_{n+1}(a), f_{n+1}(p)]_{n+1}) = [f_n(a), f_n(p)]_n$, for each n and each point $p \in A_n$.

One can check that it is possible to choose maps f_n and g_n, $n = 1, 2, \ldots$, in such a manner that $g_{n+1} \mid_{[f_{n+1}(a), f_{n+1}(p)]_{n+1}}$ is monotone, for each point $p \in A_n$.

For any positive integers n, k and points $b_1, \ldots, b_k \in A_n$ put

$$D_n(b_1, \ldots, b_k) = \bigcup_{i=1}^{k} [f_n(a), f_n(b_i)]_n;$$

so $D_n(b_1, \ldots, b_k)$ is a finite dendron and

$$g_n \mid_{D_{n+1}(b_1,\ldots,b_k)} : D_{n+1}(b_1, \ldots, b_k) \to D_n(b_1, \ldots, b_k)$$

is a monotone mapping. Put

$$D(b_1, \ldots, b_k) = \lim \operatorname{inv}(D_{m+1}(b_1, \ldots, b_k), g_m \mid_{D_{m+1}(b_1,\ldots,b_k)}, m \geq n);$$

so $D(b_1, \ldots, b_k)$ is a finite dendron in X (see for example [**11**]). Finally, put

$$J = \{D(b_1, \ldots, b_k) : k = 1, 2, \ldots, b_1, \ldots, b_k \in A_n, n = 1, 2, \ldots\}.$$

It follows that J is a family of finite dendrons which approximates X.

REFERENCES

1. C. E. Capel, *Inverse limit spaces*, Duke Math. J. **21** (1954), 233–245.
2. J. L. Cornette, *Image of a Hausdorff arc is cyclically extensible and reducible*, Trans. Amer. Math. Soc. **199** (1974), 255–267.
3. R. Engelking, *General topology*, PWN, 1977.
4. G. R. Gordh and S. Mardešić, *Characterizing local connectedness in inverse limits*, Pacific J. Math. **58** (1975), 411–417.
5. H. Hahn, *Mengentheoretische Charakterisierung der stetigen Kurven*, Sitzungsber. Akad. Wiss. Wien **123** (1914), 2433–2489.
6. J. G. Hocking and G. S. Young, *Topology*, Addison-Wesley, 1961.
7. S. Mardešić, *On the Hahn-Mazurkiewicz theorem in non-metric spaces*, Proc. Amer. Math. Soc. **11** (1960), 929–937.
8. ———, *Locally connected, ordered and chainable continua* (in Serbo-Croatian, English summary), Rad Jugoslav. Akad. Znan. Umjet. Odjel Mat. Fiz. Tehn. Nauke **319** (1961), 147–166.
9. ———, *On the Hahn-Mazurkiewicz problem in non-metric spaces*, General Topology and its Relations to Modern Analysis and Algebra II, Prague, 1966.
10. S. Mazurkiewicz, *Sur les lignes de Jordan*, Fund. Math. **1** (1920), 166–209.
11. J. van Mill and E. Wattel, *Dendrons*, Topology and Order Structures, part I, Math. Centre Tracts, vol. 142, Amsterdam, 1981, pp. 59–81.
12. J. Nikiel, *Images of arcs — a nonseparable version of the Hahn-Mazurkiewicz theorem*, Fund. Math. **129** (1988).
13. ———, *An elementary proof of a theorem of S. Mardešić*, Proc. Amer. Math. Soc. **101** (1987), 385–386.
14. ———, *A continuous partial ordering for images of arcs*, Proc. Sixth Prague Topological Symp. 1986 (to appear).
15. B. J. Pearson, *Mapping arcs and dendritic spaces onto netlike continua*, Colloq. Math. **34** (1975), 39–48.
16. L. B. Treybig, *Arcwise connectivity in continuous images of ordered compacta*, Glasnik Mat. **21** (1986), 201–211.
17. L. B. Treybig and L. E. Ward, *The Hahn-Mazurkiewicz problem*, Topology and Order Structures, part I, Math. Centre Tracts, vol. 142, Amsterdam, 1981, pp. 95–105.
18. L. E. Ward, *The Hahn-Mazurkiewicz theorem for rim-finite continua*, General Topology and Appl. **6** (1976), 183–190.
19. ———, *A generalization of the Hahn-Mazurkiewicz theorem*, Proc. Amer. Math. Soc. **58** (1976), 369–374.
20. G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloq. Publ., vol. 28, Amer. Math. Soc., Providence, R. I., 1942.
21. ———, *Cut points in general topological spaces*, Proc. Nat. Acad. Sci. U.S.A. **61** (1968), 380–387.
22. G. S. Young, *Representations of Banach spaces*, Proc. Amer. Math. Soc. **13** (1962), 667–668.

INSTITUTE OF MATHEMATICS, UNIVERSITY OF WROCLAW, PL. GRUNWALDZKI 2/4, 50-384 WROCLAW, POLAND