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COMPACTLY GENERATED SUBGROUPS AND OPEN SUBGROUPS OF LOCALLY COMPACT GROUPS

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ABSTRACT. This paper contains results of the following sort: If G is a locally compact group and H is a closed subgroup such that the coset space G/H is locally connected, then HG_0 is open in G. If G is a locally compact group such that G/G_0 is compact, then every closed subgroup of G is compactly generated if and only if G_0 has no noncompact simple factor.

This paper consists of applications and extensions of the results of the present authors in four previous papers, [**1–4**]. Most of the terminology is consistent with that of [**3**] in which there is a motivating introduction.

In §1, we consider some sufficient conditions for subgroups of the form HG_0 to be open or closed, where H is a subgroup of a locally compact group G. We show that if H is a closed subgroup of a locally compact group G such that the coset space G/H is locally connected, then GH_0 is open in G. In regard to the question of whether or not the product of two closed subgroups is closed, we show that, if G is a first countable locally compact group, and if K and H are closed subgroups of G with the property that, for some neighborhood U of e in G, $HUH \cap K \subset H$, then KH is closed, Remark 1.10. One should compare this result with [**9**, Lemma 1.15]. In §2, we discuss closed subgroups of compactly generated groups, considering conditions under which a closed subgroup of a compactly generated group is itself compactly generated. We show that if G is a locally compact group such that G/G_0 is compact, then every closed normal subgroup is compactly generated, Theorem 2.2. We also show that, if G is a locally compact group such that G/G_0 is compact, then every closed subgroup of G is compactly generated if and only if G_0 has no noncompact simple factor, Theorem 2.8. Proposition 2.1 generalizes the result of Grosser amd Moskowitz, that every closed subgroup of a compactly generated $[FC]$-group is compactly generated [**5**].

Subgroups of the form HG_0. In this section we shall consider some sufficient conditions for subgroups of the form HG_0 to be open, where H is a subgroup of the locally compact group G.

LEMMA 1.1. *If H is a closed normal subgroup of a locally compact group G such that G/H is a Lie group and $G = KG_0$, where K is a totally disconnected compact normal subgroup, such that $K \cap G_0 = e$, then HG_0 is open in G.*

PROOF. Since KH is closed $KH/H \cong K/K \cap H$ is a Lie group. Since K is totally disconnected, $K/K\cap H$ is discrete; so $K\cap H$ is open in K. Since $K\cap G_0 = e$,

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we have $KG_0/G_0 \cong K$; and $(K \cap H)G_0$ is open in $KG_0 = G$. Thus HG_0 is open in G.

LEMMA 1.2. *If H is a closed normal subgroup of a locally compact group such that G/H is a Lie group, and if G_0 contains no nontrivial compact normal subgroups, then HG_0 is open.*

PROOF. Let G_1 be an open subgroup of G such that G_1/G_0 is compact, [**7**, 2.3.1]. There is a totally disconnected compact normal subgroup K of G_1 such that $G_1 = KG_0$ [**6**]. Now $G_1/G_1 \cap H \cong G_1H/H$ which is a Lie group. Thus $(G_1 \cap H)G_0$ is open in G_1, hence in G, by Lemma 1.1. It follows that HG_0 is open in G.

The next theorem is generalized by some theorems which follow it. We state it separately as a theorem since it covers the most interesting case.

THEOREM 1.3. *If H is a closed normal subgroup of a locally compact group G such that G/H is a Lie group, then HG_0 is open.*

PROOF. Let K be maximal compact normal subgroup of G_0. Then G/K is a locally compact group with identity component G_0/K which has no nontrivial compact normal subgroups and $G/K/HK/K \cong G/HK$ is a Lie group. By Lemma 1.2, $(HK/K)(G_0/K) = HG_0/K$ is open in G/K. Consequently HG_0 is open in G.

THEOREM 1.4. *If H is a closed subgroup of a locally compact group G and G/H is locally connected, then G_0H is open.*

PROOF. We can assume that G_0 is a Lie group. To see this, we let H and G satisfy the hypothesis and let K be the compact normal subgroup of G_0 such that G_0/K is a Lie group [**7**, 4.6]. We have $G_0H/K \cong (G/K)_0(H/K), G/KH$ homeomorphic to $G/K/KH/K$, and G/KH locally connected. Now, if G_0H/K is open in G/K, then G_0H is open in G. Thus our assumption from this point on that G_0 is a Lie group is justified. We can also assume that G/G_0 is compact, by using the fact that there is an open subgroup G_1 such that G_1/G_0 is compact and employing the technique in the proof of Lemma 1.2.

A further reduction can be made as follows. By Theorem 2.6, p. 58 of [**7**], there is a compact normal subgroup K such that G/K is metric. It follows that G is finite dimensional and G/H is also. If $N = \{g \in G\colon gxH = xH \text{ for all } x \in G\}$, then N is normal in G, $N \subset H$, and G/N acts effectively on G/H is an obvious way. By the corollary of paragraph 6.3, p. 243 of [**7**], G/N is a Lie group. Thus, by Theorem 1.3, G_0N is open in G. The proof is complete since $G_0H \supset G_0N$.

If K is a compact normal subgroup of a locally compact group G such that G/K is locally connected, then G_0K/K is the identity component of G/K. Thus, G_0K is open in G and this special case does not depend on Theorem 1.4. Even so, we state the following result as a corollary for purpose of easy reference.

COROLLARY. *Let G be a locally compact group. For G/G_0 to be compact it is necessary and sufficient that there exists a compact normal subgroup K of G such that G/K is a Lie group with a finite number of components.*

PROOF. The necessity is well known, p. 175 of [**7**].

For the proof of the sufficiency, let K be a compact normal subgroup of G such that G/K is a Lie group with a finite number of components. It follows that G/G_0K

is finite since G_0K is open and G_0K/K is the identity component of G/K. Thus G/G_0 is compact.

We call a closed subgroup H of G uniform if the coset space G/H is compact.

REMARK 1.5. Let H be a uniform subgroup of a locally compact group G. If Q is a compact normal subgroup of H, then Q is contained in a compact normal subgroup of G.

PROOF. Since G/H is compact, $G = HD$ for some compact set $D = D^{-1}$. For $x \in Q$, $\{dhxh^{-1}d^{-1} \colon h \in H, d \in D\} \subset DQD$ and DQD is compact. Hence $Q \subset P(G) \cap B(G)$, where $P(G)$ and $B(G)$ are the periodic elements and bounded element of G respectively, and Q generates a compact normal subgroup of G, [**5**].

COROLLARY. *Let H be a closed pro-Lie uniform subgroup of a locally compact group G. Then there exists a compact normal subgroup P of G such that HP/P is a Lie group. In particular, if H is a compactly generated locally projective nilpotent group, we can choose P so that HP/P is a torsion free compactly generated nilpotent Lie group.*

REMARK 1.6. If G is a locally compact group, G_1 an open subgroup, and H a closed subgroup of G, then G_0H is closed if and only if $G_0H \cap G_1$ is closed.

PROOF. The "only if" part is clear.

Suppose now that $G_0H \cap G_1$ is closed, and let $F = \overline{G_0H}$. Then $F \cap G_1$ is open in F. Hence $(G_0H) \cap G_1$ is dense in $F \cap G_1$. If $G_0H \cap G_1$ is closed, then $G_0H \cap G_1 = F \cap G_1$, so $G_0H \cap G_1$ is open in F. Thus $G_0H = F = \overline{G_0H}$, and G_0H is closed.

REMARK 1.7. Let G be a locally compact group, and H a closed subgroup of G. Then G_0H is closed if for some open subgroup G_1 of G such that G_1/G_0 is compact, we have that $H_1/H_1 \cap G_0$ is compact, where $H_1 = H \cap G_1$.

PROOF. Since $H_1/H_1 \cap G_0$ is compact, H_1G_0/G_0 is compact. Hence H_1G_0 is closed, and, consequently, $HG_0 = G_0H$ is closed also.

If G is pro-Lie, then there is an open normal subgroup G_1 of G such that G_1/G_0 is compact. This is Corollary 2 of Proposition 1.6 [**3**]. In relation to this result and the corollary of Theorem 1.4 above, we have the following example of a pro-Lie group G with a closed subgroup H such that G_0H is not closed.

EXAMPLE 1.1. Let $G = \mathbf{R} \times D$, where $\mathbf{R}$ is the additive group of real numbers and D the dyadic numbers. Then $G_0 = \mathbf{R}$. Let u be the monothetic generator of D and let $H = \{(n, u^n) \colon n \text{ an integer}\}$. Then H is a closed subgroup of G but G_0H is not closed. However, G is a pro-Lie group since it is abelian and locally compact.

PROPOSITION 1.8. *Let G be a pro-Lie group, and H a closed subgroup of G. Then there exists a compact subgroup L of G such that $\overline{G_0H} = G_0HL$, L is a normal subgroup of $\overline{G_0H}$, and $\overline{G_0H}/L$ is a Lie group.*

PROOF. Since G is pro-Lie, $\overline{G_0H}$ is pro-Lie. Let K be any compact normal subgroup of G such that G/K is Lie. Then G/K contains G_0K/K as an open subgroup since G_0K/K is the identity component of G/K. Hence G_0KH/K is an open and closed subgroup of G/K, and we have $\overline{G_0H} \subset G_0HK$. If $L = \overline{G_0H} \cap K$, L is normal in G_0H and is a compact subgroup of G. We claim that G_0HL is closed and contains G_0H. Note that $\overline{G_0H}K/K$ is a Lie group and there is a continuous isomorphism of $\overline{G_0H}/\overline{G_0H} \cap K$ onto $\overline{G_0H}K/K$. Hence $\overline{G_0H}/G_0H \cap K$ is also a Lie group. By Theorem 1.4, G_0L is open in $\overline{G_0H}$, so G_0LH is open and hence closed in $\overline{G_0H}$. Now $G_0HL = G_0LH = \overline{G_0H}$; so we have proved the claim, and hence the propostion.

In general, it is very difficult to give a condition under which the product of two closed subgroups is closed; e.g. see [**11**, Theorem 2.7]. The following two remarks should be compared with Lemma 1.14 and Lemma 1.15 of [**9**].

REMARK 1.9. Let G be a first countable locally compact group, and let K be a discrete subgroup of G. If Δ denotes a subgroup of K and $N = N_G(\Delta)$ denotes the normalizer of Δ in G, then NK is closed in G.

PROOF. Let $\{n_i\gamma_i\}$ be a sequence in NK converging to $z \in G$, where $n_i \in N$ and $\gamma_i \in K$. Let δ be any element of Δ. Then $\{\gamma_i^{-1}n_i^{-1}\delta n_i\gamma_i\}$ is a sequence in K converging to $z^{-1}\delta z$. Because K is discrete, there exists i_0 such that $\gamma_i^{-1}n_i^{-1}\delta n_i\gamma_i = \gamma_{i_0}^{-1}n_{i_0}^{-1}\delta n_{i_0}\gamma_{i_0}$ for every $i \geq i_0$. This means that $\delta n_i\gamma_i\gamma_{i_0}^{-1}n_{i_0}^{-1} = n_i\gamma_i\gamma_{i_0}^{-1}n_{i_0}^{-1}\delta$ for each $i \geq i_0$. Hence $n_i\gamma_i\gamma_{i_0}^{-1}n_{i_0}^{-1} \in Z_G(\Delta)$ for each $i \geq i_0$, where $Z_G(\Delta)$ denotes the centralizer of Δ in G. Consequently, $n_{i_0}\gamma_{i_0}\gamma_i^{-1}n_i \in Z_G(\Delta) \subset N_G(\Delta) = N$ for each $i \geq i_0$. This shows that $\gamma_{i_0}\gamma_i^{-1} \in N$ for each $i \geq i_0$. For each $i \geq i_0$, let $\gamma_i\gamma_{i_0}^{-1} = x_i \in N$, then $n_i\gamma_i = n_ix_i\gamma_{i_0} = n_i'\gamma_{i_0}$, where $n_i' = n_ix_i \in N$. Since $n_i\gamma_i \to z$, we have $n_i'\gamma_{i_0} \to z$ and, hence, $n_i' \to z\gamma_{i_0}^{-1}$. Now N is closed, so $z\gamma_{i_0}^{-1} = n \in N$. Thus $z = n\gamma_{i_0} \in NK$, and NK is closed.

REMARK 1.10. Let G be a first countable locally compact group, and let K and H be closed subgroups of G with the property that, for some neighborhood Ω of e in G, $H\Omega H \cap K \subset H$. Then HK is closed in G.

PROOF. Let $\{h_i\gamma_i\}$ be a sequence in HK converging to $z \in G$. Choose a neighborhood V of e in G such that $V = V^{-1} \subset V^2 \subset \Omega$. Then there exists i_0 such that $h_i\gamma_i \in Vz$ for $i \geq i_0$. Hence for each $i \geq i_0$, there is $v_i \in V$ such that $v_ih_i\gamma_i = z$. Consequently, we have $\gamma_{i_0}^{-1}h_{i_0}^{-1}v_{i_0}^{-1}v_ih_i\gamma_i = e$, i.e. $h_{i_0}^{-1}v_{i_0}^{-1}v_ih_i = \gamma_{i_0}\gamma_i^{-1} \in K$ for each $i \geq i_0$. Since $HV^2H \cap K \subset H$, $\gamma_{i_0}\gamma_i^{-1} \in H$ for each $i \geq i_0$. Thus for each $i \geq i_0$, $\gamma_i = h_i'\gamma_{i_0}$ for some $h_i' \in H$. This implies that $h_ih_i'\gamma_{i_0} = h_i\gamma_i \to z$, or, i.e. $h_ih_i' \to z\gamma_{i_0}^{-1}$, and we have that $z\gamma_0^{-1} \in H$. Hence $z \in H\gamma_{i_0} \subset HK$, and the proof is completed.

2. Closed subgroups of compactly generated groups. It is well known that not every closed subgroup of a compactly generated group is compactly generated. In [**8**] it is proved that every closed subgroup of a compactly generated nilpotent group is compactly generated; and it is shown in [**5**] that every closed subgroup of a compactly generated $[FC]^-$-group is compactly generated. In this section we shall give several sufficient conditions under which a closed subgroup of a compactly generated group is itself compactly generated. One of our results extends the result stated above for $[FC]^-$-groups. By "compactly generated" we mean generated by a compact neighborhood of the identity. In particular, compactly generated groups are locally compact.

It is clear that, if G is a locally compact group and if K is a closed uniform subgroup of G, then K is compactly generated if and only if G is compactly generated.

PROPOSITION 2.1. *Suppose that H is a closed normal subgroup of a locally compact group G, and that every closed subgroup of G/H and every closed subgroup* *of H is compactly generated. If F is a closed subgroup of G such that FH is closed, then F is compactly generated.*

PROOF. Since both G/H and H are compactly generated, G is compactly generated, hence G is σ-compact. As a closed subgroup of a σ-compact group, F is σ-compact. If FH is closed, FH/H is compactly generated by assumption. Hence $F/F \cap H$ is also compactly generated since $F/F \cap H \cong FH/H$. Since $F \cap H$ is compactly generated, F is compactly generated as desired.

COROLLARY 1. *If K is a compact normal subgroup of a locally compact group G, and if every closed subgroup of G/K is compactly generated, then every closed subgroup of G is compactly generated.*

COROLLARY 2 [**5**]. *Every closed subgroup of a compactly generated $[FC]^-$-group is compactly generated.*

PROOF. There exists a compact normal subgroup K such that $G/K \cong V \times Z^l$, where V is a vector group and l a positive integer. Thus every closed subgroup of G/K is compactly generated. The conclusion now follows from Corollary 1 above.

COROLLARY 3. *Let G be a compactly generated group. If there exists a compact normal subgroup K such that G/K is a connected solvable group, then every closed subgroup of G is compactly generated.*

PROOF. Every closed subgroup of a connected solvable Lie group is compactly generated.

THEOREM 2.2. *Let G be a locally compact group such that G/G_0 is compact. Then every closed normal subgroup is compactly generated.*

PROOF. It is clear that G is compactly generated. Let F be a closed normal subgroup of G. Let Q be a maximal compact normal subgroup of G such that G/Q is a Lie group. Note that, if we can show that FQ/Q is compactly generated, then F will also be compactly generated. Hence we may assume that G is a Lie group. With this assumption, G/G_0 is a finite Lie group. Hence $F/F \cap G_0 \cong FG_0/G_0$ is finite. Thus we may further assume that G is connected. Now F_0 is normal in G, G/F_0 is connected and F/F_0 is a totally disconnected normal subgroup. Hence F/F_0 is central in G/F_0. Therefore F/F_0 is finitely generated. This implies that F is compactly generated.

COROLLARY. *The center of a locally compact group G such that G/G_0 is compact is compactly generated.*

PROPOSITION 2.3. *Let G be a compactly generated pro-Lie group and H a closed subgroup such that G/H has a finite invariant measure. Then H is compactly generated.*

PROOF. Let K be any compact normal subgroup of G such that G/K is a Lie group. Then $G/K/HK/K$ has a finite invariant measure. By Theorem 4.3 of [**9**], HK/K is compactly generated. Hence H is compactly generated.

Recall that the radical of a locally compact group G is a maximal normal solvable connected subgroup of G, and that every subgroup of a Lie group admits a unique maximal normal solvable subgroup [**9**, Corollary 8.6, p. 142].

LEMMA 2.4. *Let G be an analytic group such that G/R is compact, where R is the radical of G, and let H be any closed subgroup of G. If H' denotes the unique maximal normal solvable subgroup of H, then H/H' is compact.*

PROOF. Let $F = \overline{RH}$. Then $F = R(F)H$, [**12**], where $R(F)$ denotes the radical of F. Since G/R is compact, and $R \subset R(F)$, it follows that $G/R(F)$ is compact. This implies that $F/R(F)$ is compact, since F is closed in G. This means that $H/H \cap R(F) = F/R(F)$ is compact. Now $R(F) \cap H$ is a normal solvable subgroup of H. Hence $R(F) \cap H \subset H' \subset H$, and H/H' is compact.

LEMMA 2.5. *Let G be an analytic solvable group, H a closed subgroup, and H_0 the identity component of H. Then every subgroup of H/H_0 is finitely generated.*

PROOF. This is Corollary 3.9, p. 54 of [**9**].

THEOREM 2.6. *If G is a connected locally compact group such that G/R is compact, where R is the radical of G, then every closed subgroup of G is compactly generated.*

PROOF. First note that we may assume that G is an analytic group. To see this let K be a maximal compact normal subgroup of G such that G/K is a Lie group. Then it is easy to see that, in order to show every closed subgroup of G is compactly generated, it is sufficient to show every closed subgroup of G/K is compactly generated. This justifies our assumption that G is analytic.

Let H be a closed subgroup of G. Then H/H' is compact by Lemma 2.4. let $F = \overline{RH}$. Then, by the proof of Lemma 2.4, $R(F) \cap H \subset H' \subset H$, and $H/R(F) \cap H$ is compact. If $\widetilde{H} = R(F) \cap H$, then $\widetilde{H}$ is a subgroup of a connected solvable group $R(F)$. Now $\widetilde{H}/(\widetilde{H})_0$ is finitely generated by Lemma 2.5, and $(\widetilde{H})_0$ is connected. Hence $\widetilde{H}$ is compactly generated. Consequently, H is compactly generated.

LEMMA 2.7. *Let G be a locally compact group such that G/G_0 is compact. If Q is the maximal compact normal subgroup of G such that G/Q is a Lie group, then every closed subgroup of G is compactly generated if and only if every closed subgroup of G_0Q is compactly generated.*

PROOF. It is clear that, if every closed subgroup of G is compactly generated, then every closed subgroup of G_0Q is compactly generated.

Conversely, assume that every closed subgroup of G_0Q is compactly generated, and let L be a closed subgroup of G. Since G/Q is a Lie group, Theorem 1.3 implies that G_0Q is open in G. Hence LG_0Q is open and closed in G. Now G/G_0Q is compact. Hence L is compactly generated by Proposition 2.1.

THEOREM 2.8. *Let G be a locally compact group such that G/G_0 is compact. Then every closed subgroup of G is compactly generated if and only if G_0 has no noncompact simple factor.*

PROOF. Suppose that G_0 has no noncompact simple factor. Then $G_0/R(G_0)$ is compact, and hence every closed subgroup of G_0 is compactly generated by Thoerem 2.6. Let Q be the maximal compact normal subgroup of G. It follows that every closed subgroup of G_0Q, and hence every closed subgroup of G is compactly generated.

For the converse, we note first that, since $G_0 \cap Q$ is the maximal compact normal subgroup of G_0 such that $G_0/G_0 \cap Q$ is a Lie group, G_0Q/Q is a Lie group. Hence we may assume that G_0 is an analytic group. Suppose G_0 has a noncompact analytic simple subgroup S. Using the adjoint representation, we may assume that S is a linear group. By Theorem 14.1 of [**9**], S contains a lattice Γ. Then Γ contains a nonabelian free group Γ'. For otherwise Γ contains a solvable subgroup Δ of finite index, [**10**]. By the density theorem of Borel [**9**, Chapter V], Γ is algebraically dense in S. Thus S is solvable. This contradiction completes the proof.

A locally compact group is a generalized $[FC]$-group, [**1**], if $G = A_1 \supset A_2 \supset \cdots \supset A_n \supset A_{n+1} = e$, where each A_i is a closed normal subgroup of G such that A_i/A_{i+1} is a compactly generated $[FC]^-$-group. We note that every analytic generalized $[FC]^-$-group has no nontrivial noncompact simple factor.

LEMMA 2.9. *Let H be a closed normal subgroup of a locally compact group G such that G/H is compact, and let every closed subgroup of H be compactly generated. If G_1 is any open subgroup of G such that G_1/G_0 is compact, then every closed subgroup of G_1 is compactly generated.*

PROOF. We may assume that $G_1 = G$. To see this we first note that G_1H is an open subgroup and that G_1H/H is compact. Since G_1 is open we have the natural homeomorphism of $G_1/G_1 \cap H$ onto G_1H/H. If we let $H_1 = G_1 \cap H$, H_1 is a closed normal subgroup of G_1, G_1/H_1 is compact and every closed subgroup of H_1 is compactly generated. Thus our assumption is justified.

Furthermore we may assume that G is analytic since G/G_0 is compact and since factoring out the maximal compact normal subgroup of G results in a Lie group. Now every closed subgroup of H is compactly generated. Hence, by Theorem 2.8, H_0 has no noncompact simple factor. Let S be a noncompact simple factor of G. Then $S \cap H_0$ is normal in S. But S has no proper normal subgroup unless it is central in S. $\overline{SH_0}/H_0$ has a noncompact simple factor. If $G' = G/H_0$ and $H' = H/H_0$, H' is central in G', and $G'/H \cong G/H$ is compact. Hence G' does not contain any nontrivial noncompact simple factor, and, consequently, SH_0/H_0 is trivial, and the proof is completed.

PROPOSITION 2.10. *Suppose that H is a closed uniform normal subgroup of a locally compact group G such that every closed subgroup of H is compactly generated, and H normalizes some open subgroup G_1 of G, where G_1/G_0 is compact. Then every closed subgroup of G is compactly generated.*

PROOF. Let F be a closed subgroup of G. By the lemma above, every closed subgroup of G_1 is compactly generated. Since G_1H is open in G, G/G_1H is finite. Let $F_1 = F \cap G_1H$. We have $F_1/F_1 \cap G_1 \cong F_1G_1/G_1 \subset G_1H/G_1 \cong H/H \cap G_1$. Thus $F_1/F_1 \cap G_1$, is compactly generated, being topologically isomorphic to a closed subgroup of $H/H \cap G_1$. Thus F_1 is compactly generated since $F_1 \cap G_1$ is. Now F is compactly generated since $F/F_1 = F/F \cap G_1H \cong F(G_1H)/G_1H$ is finite.

THEOREM 2.11. *Let H be a closed uniform normal subgroup of a locally compact group G such that every closed subgroup of H is compactly generated. If F is a closed subgroup of G which normalizes an open subgroup G_1 of G, where G_1/G_0 is compact, then F is compactly generated.*

PROOF. Since G_1F is open in G, G_1FH/H is compact. Now $G_1F/G_1F \cap H \cong G_1FH/H$, and $G_1F \cap H$ is compactly generated. Hence G_1F is compactly generated. Also $F/F \cap G_1 \cong G_1F/G_1$, and $F/F \cap G_1$ is compactly generated. Furthermore, $F \cap G_1$ is compactly generated by Lemma 2.9. Hence F is compactly generated.

COROLLARY 1. *Let H be a closed uniform normal subgroup of a locally compact group G such that every closed subgroup of H is compactly generated. Then every closed normal subgroup of G is compactly generated.*

COROLLARY 2. *If, in addition, H is assumed to be pro-Lie in Corollary* 1 *then every closed subgroup of G is compactly generated.*

REMARK. If H is a compactly generated projective nilpotent subgroup of G such that G/H is compact, then every closed subgroup of H is compactly generated, and hence every closed subgroup of G is compactly generated.

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