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COMMON FIXED POINTS FOR COMMUTING AND COMPATIBLE MAPS ON COMPACTA

GERALD JUNGCK

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ABSTRACT. Compatible maps—a generalization of commuting maps—are characterized in terms of coincidence points, and common fixed point theorems for compatible maps and commuting maps on compact metric spaces are obtained.

1. Introduction. Maps $f, g\colon X \to X$ are said to commute iff $fg = gf$. The concept of commuting maps has proven useful for generalizing in the context of metric space fixed point theory (see, e.g., [**1**, **2**, **4**–**11**, **15**, **16**, **17**]). Recently a less restrictive concept called compatibility was introduced in [**12**] and promoted as a means to more comprehensive results.

Now any two self-maps f and g of a set X commute on the set $\{x \in X\colon f(x) = g(x) = x\}$ of common fixed points of f and g. As we shall show, if f and g are continuous and X is compact metric, f and g are compatible iff they commute on the set $\{x \in X\colon f(x) = g(x)\}$ of coincidence points of f and g. The purpose of this note is to consider and to emulate the relative merits of compatibility and commutativity of maps in the setting of compact metric spaces. We shall do so by proving three fixed point theorems which extend results by Fisher, Leader, Das and Debata, and the author.

As to notation, we let R denote the reals with usual topology, N the set of natural numbers, and $N_0 = N \cup \{0\}$. If $f\colon X \to X$, C_f denotes the set of all maps $g\colon X \to X$ which commute with f, and we shall write fx for $f(x)$ when convenient.

2. Compatible maps.

DEFINITION 2.1 [**12**]. Self-maps f and g of a metric space (X, d) are compatible iff $\lim_n d(fgx_n, gfx_n) = 0$ when $\{x_n\}$ is a sequence such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X.

Thus, if $d(fgx, gfx) \to 0$ as $d(fx, gx) \to 0$, then f and g are compatible. So if f and g commute they are obviously compatible. On the other hand, let $fx = 5x^3$ and $gx = 2x^3$ for x in R. Then $|gx - fx| = 3|x|^3 \to 0$ iff $x \to 0$, and $|fgx - gfx| = 210|x|^9 \to 0$ iff $x \to 0$. So f and g are compatible although they do not commute. In fact, f and g are not even weakly commutative. Sessa defined self-maps f and g of (X, d) to be weakly commuting iff $d(fgx, gfx) \le d(fx, gx)$ for x in X. Clearly, commuting maps are weakly commuting and weakly commuting maps are compatible, but neither implication is reversible as examples in [**18**] and the above example (respectively) show.

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We now prepare to simplify the criterion given in Definition 2.1. by citing Proposition 2.2(1) of [**12**] which states that if self-maps f and g of a metric space are compatible then $fgx = gfx$ when $fx = gx$. An example to follow shows that the converse is not true in general. But upon noting that a mapping $f\colon X \to Y$ between topological spaces is proper iff $f^{-1}(C)$ is compact in X when C is compact in Y, we can say:

THEOREM 2.2. *Let f and g be continuous self-maps of a metric space (X, d). If f is a proper map, then f and g are compatible iff $fx = gx$ implies $fgx = gfx$.*

PROOF. The necessity of the condition follows from Proposition 2.2(1) of [**12**], or can be easily proved by supposing that $f(x) = g(x)$ and considering the sequence $\{x_n\}$ where $x_n = x$ for $n \in N$. To prove sufficiency let $\{x_n\}$ be a sequence in X and suppose that

$$\lim_n fx_n = \lim_n gx_n = t, \quad \text{for some } t \in X. \tag{1}$$

Then $S = \{fx_n \colon n \in N\} \cup \{t\}$ is compact, so that $f^{-1}(S)$ is compact since f is proper. Consequently, $\{x_n\}$ has a subsequence $\{x_{k_n}\}$ which converges to an element c of X. Since f and g are continuous, $fx_{k_n} \to fc$ and $gx_{k_n} \to gc$. But then (1) implies

$$fx_n, gx_n \to t = f(c) = g(c), \tag{2}$$

and $fgc = gfc$ by hypothesis. Therefore, since $fgx_n \to fgc$ and $gfx_n \to gfc$ by (2) and the continuity of f and g, $d(fgx_n, gfx_n) \to 0$ as desired. □

EXAMPLE 2.4. Let $X = R$, $g(x) = 2 - x^2$ and $f(x) = x^2$. f and g are both continuous and proper. If $f(x) = g(x)$, then $x = \pm 1$. $gf(\pm 1) = 1 = fg(\pm 1)$ so that f and g are compatible; but they are not weakly commuting (let $x = 3$).

Since continuous self-maps of compact metric spaces are very proper, we have the following.

COROLLARY 2.3. *Two continuous self-maps of a compact metric space are compatible iff they commute on their set of coincidence points.*

The following example, referred to above, demonstrates the essential role played by "proper maps" in ensuring compatibility.

EXAMPLE 2.5. Let $f, g\colon [0, \infty) \to [0, \infty)$ be defined by $f(x) = x(x+1)^{-1}$ and $g(x) = (\underline{x}$ for $x < 1$ and $\underline{1}$ for $x \geq 1)$. Now f and g are both continuous but neither is proper. Also, $fx = gx$ implies $x = 0$, and $f(0) = g(0) = 0 = fg(0) = gf(0)$, so that the conditions of Theorem (2.2) except for that of being proper are met. But if $x_n = n$ for $n \in N$, then $\lim_n fx_n = \lim_n gx_n = 1$, whereas $\lim_n d(fgx_n, gfx_n) = d(1/2, 1) \neq 0$; thus f and g are not compatible.

COROLLARY 2.6. *Suppose that f and g are continuous self-maps of a metric space and that f is proper. If $fx = gx$ implies $x = fx$, then f and g are compatible.*

PROOF. If $fx = gx$, then $x = fx = gx$ and hence $fgx = gfx$. f and g are therefore compatible by Theorem 2.2. □

Note that the condition of Corollary 2.6 is sufficient but not necessary since the functions f and g of Example 2.4 are continuous, proper, and compatible, but $f(-1) = g(-1) = 1$

The next result tells us that nice functions on nice spaces are compatible with lots of functions.

COROLLARY 2.7. *Let M be a convex subset of a normed linear space and let $f\colon M \to M$ be proper and continuous. If $s\colon M \to [0,1)$ is continuous and if $g_s(x) = (1 - s(x))x + s(x)f(x)$ for x in M, then f and g_s are compatible.*

PROOF. Since M is convex, $g_s\colon M \to M$, and g_s is continuous since s and f are. Moreover, if $g_s(x) = f(x)$, then $(1 - s(x))f(x) = (1 - s(x))x$ so that $f(x) = x$ since $s(x) \neq 1$; i.e., f and g_s are compatible by Corollary 2.6. □

Functions of the form g_s with s constant give rise to iteration processes which produce sequences converging to fixed points of f (see e.g., [**3**]). We refer the reader to [**12**] for further properties of compatible maps and for other examples which show that compatible maps need not be weakly commuting (and hence not commutative). Note also that Sessa has extended a variety of fixed theorems by substituting weak commutativity for commutativity; we cite [**18** and **19**] as examples.

3. A fixed point theorem for compatible maps. We appeal to the following generalization of a theorem of S. P. and S. L. Singh [**20**] to prove our next result.

THEOREM 3.1 [**13**]. *Let A, B, S and T be self-maps of a complete metric space (X, d). Suppose that S and T are continuous, the pairs A, S, and B, T are compatible pairs, and that $A(X) \subset T(X)$ and $B(X) \subset S(X)$. If there exists $r \in (0,1)$ such that $d(Ax, By) \leq r \max(Mxy)$ for x, y in X, where*

$$(1)\qquad Mxy = \{d(Sx, Ty), d(Ax, Sx), d(By, Ty), \tfrac{1}{2}(d(Ax, Ty) + d(Sx, By))\},$$

then there is a unique point z in X such that $Az = Bz = Sz = Tz = z$.

The following theorem generalizes Theorem 1 of Fisher [**7**] by requiring compatibility in lieu of commutativity and replacing the terms $\frac{1}{2}\, d(Ax, Ty), \frac{1}{2}\, d(Sx, By)$ by $\frac{1}{2}(d(Ax, Ty) + d(Sx, By))$ in Mxy.

THEOREM 3.2. *Let A, B, S, T be continuous self-maps of a compact metric space (X, d) with $A(X) \subset T(X)$ and $B(X) \subset S(X)$. If A, S and B, T are compatible pairs and $d(Ax, By) < \max(Mxy)$ (see (1)) when $\max(Mxy) > 0$, then A, B, S, and T have a unique common fixed point.*

PROOF. We assert $\max(Mxy) = 0$ for some pair x, y. Otherwise, the function $hxy = d(Ax, By)/\max(Mxy)$ is continuous and satisfies $hxy < 1$ on $X \times X$. Since $X \times X$ is compact, there exist $c, d \in X$ such that $hxy \leq r = hcd < 1$ for $x, y \in X$. Consequently, $d(Ax, By) \leq r \max(Mxy)$ on X with $r < 1$, so by Theorem 3.1, $Az = Bz = Sz = Tz = z$ for some z. We have the contradiction, $\max(Mzz) = 0$ and $\max(Mzz) > 0$.

Since $\max(Mxy) = 0$ for some $x, y \in X$, (1) implies

$$(2)\qquad Sx = Ax = Ty = By \text{ and thus } SBy = SAx \text{ and } ASx = ABy.$$

Since A and S are compatible, $Ax = Sx$ in (2) implies that $SAx = ASx$ and therefore $SBy = ABy$. We now prove that $SBy = By$ so that By is a common fixed point of A and$S$.

For if $SBy \neq By$, $\max(MByy) > 0$ by (1), so that by hypothesis

$$d(ABy, By) < \max\{d(SBy, Ty), d(SBy, ABy), d(By, Ty),\\ \tfrac{1}{2}(d(ABy, Ty) + d(SBy, By))\}.$$

Then (2) and the fact that $SBy = ABy$ imply

$$d(ABy, By) < \max\{d(ABy, By), 0, 0, \tfrac{1}{2}(d(ABy, By) + d(ABy, By))\},$$

which yields the contradiction: $d(ABy, By) < d(ABy, By)$.

We thus have $Av = Sv = v$, with $v = By$. Similarly, there exists $w \in X$ such that $Bw = Tw = w$. Moreover, $v = w$. If not, (1) and the hypothesis imply

$$d(v, w) < \max\{d(v, w), 0, 0, \tfrac{1}{2}(d(v, w) + d(v, w))\} = d(v, w);$$

again, a contradiction. We conclude that $v = w$ is the common fixed point of A, B, S, and T. In like manner, "uniqueness" follows immediately. □

The following example verifies that Theorem 3.2 does indeed generalize Theorem 1 of Fisher referred to above.

EXAMPLE 3.3. Let $X = [0, 1]$ and $d(x, y) = |x - y|$. Define $Sx = x^{1/2}$, $Tx = x^{1/2}/2$, $Ax = x^{1/2}/4$, and $Bx = x^{1/2}/8$. A, B, S, and T are continuous, and $A(X) = [0, \frac{1}{4}] \subset [0, \frac{1}{2}] = T(X)$; similarly, $B(X) \subset S(X)$. A and S are compatible by Corollary 2.3 since X is compact and $Ax = Sx$ implies $x = 0 = AS(0) = SA(0)$. Likewise, B and T are compatible. Moreover, $d(Ax, By) = \frac{1}{4}(Sx, Ty) \leq \frac{1}{4}\max(Mxy)$, and the hypothesis of Theorem 3.2 is satisfied. However, A and S are not weakly commutative—and hence not commutative—since $|Ax - Sx| = 3x^{1/4}|SAx - ASx|$, so that $|Ax - Sx| < |SAx - ASx|$ if $x < 1/81$. The hypothesis of Fisher's Theorem 1 is therefore not satisfied.

To better appreciate how "tight" the hypothesis of Theorem 3.2 might be and to better understand the relative roles of commutativity and compatibility, consider the following.

THEOREM 3.4 (FISHER [6]). *Let A, B, S, and T be mappings of a compact metric space (X, d) into itself satisfying*

(3) $$d(Ax, By) < \max\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), d(Sx, By), d(Ty, Ax)\}$$

for all x, y in X for which the right-hand side of (3) is positive. If A and B commute, if S and T commute with AB and if AB is continuous, then A, B, S, and T have a unique common fixed point.

Example 6 by Sessa [19] shows that Theorem 3.4 is false even if the only change in the hypothesis is to permit S and AB to be a weakly commuting pair.

4. Fixed point theorems for commuting maps. The proofs of our two remaining theorems appeal to the following.

PROPOSITION 4.1. *Let f and g be commuting self-maps of a compact metric space (X, d) such that gf is continuous. If $A = \bigcap_{n=1}^{\infty}(gf)^n(X)$, then*
(i) *$h(A) \subset A$ for $h \in C_{gf}$,*
(ii) *$A = f(A) = g(A) \neq \emptyset$, and*
(iii) *A is compact.*

PROOF. It is well known (see e.g., [14]) that A is not empty, that A is compact, and that $gf(A) = A$. If $h \in C_{gf}$, we can write

$$h(A) \subset \bigcap_{n=1}^{\infty} h(gf)^n(X) = \bigcap_{n=1}^{\infty}(gf)^n(h(X)) \subset \bigcap_{n=1}^{\infty}(gf)^n(X) = A.$$

Specifically, $g(A) \subset A$ and $f(A) \subset A$. Thus $A = gf(A) \subset g(A) \subset A$, so $g(A) = A$. Similarly, $f(A) = A$. □

In our next result we use the standard notation $\operatorname{diam}(S) = \sup\{d(x,y)\colon x, y \in S\}$ if S is a subset of a metric space (X, d).

THEOREM 4.2. *Let f and g be commuting self-maps of a compact metric space (X,d) such that gf is continuous. If*

(1) $fx \neq gy$ implies $d(fx, gy) < \operatorname{diam}\{h(z)\colon z \in \{x,y\}$ and $h \in C_{gf}\}$, then there is a unique point a in X such that $a = fa = ga$. In fact, $a = ha$ for all $h \in C_{gf}$.

PROOF. Let A be as in Proposition 4.1 so that (i), (ii), and (iii) of Proposition 4.1 hold. We assert that $A = \{a\}$ for some a in X. Otherwise $\operatorname{diam}(A) > 0$, so by compactness there exist distinct $u, v \in A$ such that $d(u,v) = \operatorname{diam}(A)$. By (ii), we can find $x, y \in A$ such that $u = fx$ and $v = gy$; i.e., $d(fx, gy) = \operatorname{diam}(A)$. Since $fx \neq gy$, (1) implies

$$\operatorname{diam}(A) = d(fx, gy) < d(h_1 z_1, h_2 z_2) \tag{2}$$

for some $h_i \in C_{gf}$ and $z_i \in \{x, y\}$ $(i = 1, 2)$. Since $z_i \in A$, (i) of Proposition 4.1 implies that $h_i z_i \in A$ for $i = 1, 2$; consequently, (2) yields the contradiction, $\operatorname{diam}(A) < \operatorname{diam}(A)$.

Thus $A = \{a\}$ for some a in X. Then (i) implies that $a = ha$ for h in C_{gf}; in particular, $a = fa = ga$. Now if $c = fc = gc$, $gfc = c$ and thus $(gf)^n c = c$ for $n \in N$; i.e., $c \in A = \{a\}$. Hence, a is the only common fixed point of f and g. □

Clearly, Theorem 3.4 follows from Theorem 4.2. For suppose that $Ax \neq By$. Then the right member of (3) in Theorem 3.4 is positive and therefore the inequality (3) holds. But since $S, T \in C_{AB}$ by hypothesis, (1) of Theorem 4.2 with $A = f$ and $B = g$ holds.

Observe also that since A in the above proof is shown to be a singleton, Leader's Theorem 1 (1° and 7°) in [**14**] assures us that $(gf)^n(x) \to a$ uniformly for all $x \in X$. Thus a is a "uniformly contractive" point for gf, but it need not be for both f and g as examples show.

Note also that in light of the above comments, the theorem by Das and Debata [**4**] follows from Theorem 4.2, and the following corollary extends Corollary 2 of Leader [**14**].

COROLLARY 4.3. *If f is a continuous self-map of a compact metric space (X,d) such that for some $r, s \in N_0$,*

$$f^r x \neq f^s y \text{ implies } d(f^r x, f^s y) < \operatorname{diam}\{hz\colon z \in \{x,y\} \,\&\, h \in C_f\}, \tag{2}$$

then f has a uniformly contractive fixed point.

PROOF. By the proof of Theorem 4.2 with $f = f^r$ and $g = f^s$, $\bigcap_{n=1}^{\infty} (f^r f^s)^n(X) = A = \{a\}$, a singleton. But $\bigcap_{n=1}^{\infty} f^n(X) \subset \bigcap_{n=1}^{\infty} f^{pn}(X)$ for any $p \in N$. Thus $\bigcap_{n=1}^{\infty} f^n(X) = \{a\}$, and the conclusion obtains by Leader's Theorem 1 (1° and 7°). □

Note. To appreciate the scope of Theorem 4.2 and hence of Corollary 4.3, observe that the functions $h \in C_{gf}$ in the right member of (1) include all functions of the form F^n with $n \in N_0$ and $F = f, g, gf$, or any function in $C_f \cap C_g$. Our final result is a cousin to Theorem 4.2, but "reverses" the inequality in (1).

THEOREM 4.4. *Let f and g be continuous commuting self-maps of a compact metric space (X, d). If*

$$(4) \qquad fx \neq gy \text{ implies } d(hx, hy) < d(fx, gy) \text{ for some } h \in C_f \cap C_g,$$

then at least one of f or g has a fixed point.

PROOF. As above we let $A = \bigcap_{n=1}^{\infty} (gf)^n(X)$, so that (i)–(iii) of Proposition 4.1 hold, noting that $C_f \cap C_g \subset C_{gf}$. Since f and g are continuous and A is compact, there exist, $a, b \in A$ such that

$$(5) \qquad d(a, f(a)) \leq d(x, f(x)) \quad \text{and} \quad d(b, g(b)) \leq d(x, g(x))$$

for $x \in A$. We assume without loss of generality that

$$(6) \qquad d(a, f(a)) \leq d(b, g(b)).$$

Since $g(A) = A$ by (ii), $g(c) = a$ for some $c \in A$. But then, if $a \neq f(a)$, $g(c) \neq f(g(c))$, and (4) yields $h \in C_f \cap C_g$ such that $d(h(c), h(g(c))) < d(g(c), f(g(c))) = d(a, f(a))$. Consequently,

$$(7) \qquad d(h(c), g(h(c))) < d(b, g(b))$$

by (6) since $h \in C_g$. But since $h \in C_f \cap C_g$, (i) implies that $h(c) \in A$ and (7) therefore contradicts the right member of (5).

We conclude that the assumption $a \neq f(a)$ is false. □

The following example reveals that not both f and g of Theorem 4.4 need have a fixed point and that the fixed point may not be unique.

EXAMPLE 4.4. Let $X = \{0, 1\}$, $d(x, y) = |x - y|$, $h = g = i$—the identity map, and define f by $f(0) = 1$, $f(1) = 0$. Then f and g are continuous commuting maps of a compact metric space into itself and $h \in C_f \cap G_g$. Moreover, $f(x) \neq g(y)$ implies $x = y$ or $i(x) = i(y) = h(x) = h(y)$, so $0 = d(h(x), h(y)) < d(f(x), g(y))$, and (4) holds. And g has two fixed points while f has none.

We conclude by observing that the sufficiency portion of Corollary 2.3 in [**9**] is a special case of Theorem 4.4 with $f = g$.

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DEPARTMENT OF MATHEMATICS, BRADLEY UNIVERSITY, PEORIA, ILLINOIS 61625