THE VIRTUAL Z-REPRESENTABILITY OF CERTAIN 3-MANIFOLD GROUPS

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(Communicated by Haynes R. Miller)

ABSTRACT. We use results on the cohomology of principal congruence subgroups of $\text{PSL}_2(\mathbb{Z}[\omega])$, $\omega^2 + \omega + 1 = 0$, to prove the existence of a large class of closed, orientable 3-manifolds with virtually $\mathbb{Z}$-representable fundamental groups. In particular, these manifolds have finite covers with positive first Betti number.

1. Let $M$ be a compact 3-manifold. We say that $M$ has a virtually $\mathbb{Z}$-representable fundamental group if some finite index subgroup $G \leq \pi_1(M)$ maps epimorphically to $\mathbb{Z}$, or equivalently, if $M$ has a finite sheeted cover $\tilde{M}$ with rank $H_1(\tilde{M}) \geq 1$. If $M$ is compact, orientable, and irreducible, then the virtual $\mathbb{Z}$-representability of $\pi_1(M)$ implies that the above cover $\tilde{M}$ is a Haken manifold (cf. [H1]). Waldhausen has conjectured that a closed, orientable, irreducible 3-manifold $M$ with infinite fundamental group is virtually Haken i.e. is finitely covered by a Haken manifold. Hence a stronger version of this conjecture is that every such $M$ has a virtually $\mathbb{Z}$-representable fundamental group.

In this paper we prove the existence of a large class of closed, orientable 3-manifolds with virtually $\mathbb{Z}$-representable fundamental groups. Let $M$ be a closed, orientable 3-manifold. Then $M$ can be realized as a branched cover of $S^3$, branched over the figure eight knot, $K$ [HLM]. We prove:

THEOREM. Let $M$ be a closed, orientable 3-manifold which is a branched cover of $S^3$, branched over the figure eight knot with all branching indices divisible by a common integer $n \geq 5$. Then $M$ has a virtually $\mathbb{Z}$-representable fundamental group.

Note. Using different methods, Hempel [H2] has proved this result in the case where the branching indices are all equal to an odd integer $n \geq 3$.

In [B] we showed that for $n \geq 5$ the $n$-fold branched cyclic covers of $S^3$ branched over the figure eight knot had virtually $\mathbb{Z}$-representable fundamental groups. The above theorem generalizes this result.

2. Let $M \rightarrow S^3$ be branched over the figure eight knot, $K$, with all branching indices divisible by a common integer $n \geq 5$, and let $X \rightarrow N$ be the associated unbranched cover obtained by removing an open tubular neighborhood of the branch set $K$ and its inverse image (thus $\partial X$, $\partial N$ are disjoint unions of tori).
We will use the fact that $N = S_3 \setminus K$ has an arithmetic hyperbolic structure. Specifically

$$\pi_1(N) = \langle x, y \mid (x^{-1}yxy^{-1})x(x^{-1}yxy^{-1})^{-1} = y \rangle$$

and there is a discrete, faithful representation $\pi_1(N) \to PSL_2(\mathbb{Z}[\omega])$, where $\omega^2 + \omega + 1 = 0$, given by

$$x \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad y \mapsto \begin{bmatrix} 1 & 0 \\ -\omega & 1 \end{bmatrix}$$

(cf. [R]). Denote by $\Gamma_K$ (resp. $\Gamma$) the image of $\pi_1(N)$ (resp. $\pi_1(X) \subset \pi_1(N)$) in $PSL_2(\mathbb{Z}[\omega])$ under this representation.

A meridian loop $\mu$ in $\partial N$ can be chosen so that $[\mu] = x$. In each component of $\partial X$ there is a loop $\alpha_i$ that projects $r_i$ to 1 onto $\mu$, where $r_i$ is the corresponding branching index in the branched cover $M \to S^3$. Hence it follows that the loop $\alpha_i$ corresponds to a homotopy class in $\pi_1(X)$ represented in $\Gamma$ by an element of the form

$$R_i \begin{bmatrix} 1 & r_i \\ 0 & 1 \end{bmatrix} R_i^{-1},$$

where $R_i \in \Gamma_K$.

\section{Consider the regular cover $Y \to X$ corresponding to the normal subgroup $\Gamma \cap \Gamma(n) \subset \Gamma$, where $\Gamma(n) \subset PSL_2(\mathbb{Z}[\omega])$ is the $n$-principal congruence subgroup (cf. [S]).}

**Lemma 1.** The loops $\alpha_i$ in $\partial X$ lift to loops in the boundary tori of $Y$ (hence these lifted loops project homeomorphically to the $\alpha_i$).

**Proof.** The loops $\alpha_i$ in $\pi_1(X)$ are represented in $\Gamma$ by elements of the form

$$R_i \begin{bmatrix} 1 & r_i \\ 0 & 1 \end{bmatrix} R_i^{-1}, \quad R_i \in \Gamma_K,$$

which are also in $\Gamma \cap \Gamma(n)$ since

$$\begin{bmatrix} 1 & r_i \\ 0 & 1 \end{bmatrix} \in \Gamma(n)$$

($n \mid r_i$ by assumption) and $\Gamma(n)$ is normal in $PSL_2(\mathbb{Z}[\omega])$. \qed

Now in each component of $\partial Y$ covering the component of $\partial X$ containing $\alpha_i$, choose one lift $\beta_{ij}$ of $\alpha_i$. Then it follows that

**Lemma 2.** The cover $Y \to X$ extends to a regular (unbranched) cover $\tilde{M} \to M$ by performing Dehn filling on $X$ and on $Y$ with respect to the loops $\{\alpha_i\}$ in $\partial X$ and $\{\beta_{ij}\}$ in $\partial Y$.

By Dehn filling on a 3-manifold $P$ with respect to a loop in a boundary torus we mean attaching a solid torus to $\partial P$ so that this loop bounds a meridional disk in the solid torus.

\section{We complete the proof of our theorem by showing that the cover $\tilde{M} \to M$ constructed in §3 satisfies rank $H_1(\tilde{M}) \geq 1$. To see this, consider the cover $Y \to X$ from which we obtained $\tilde{M} \to M$ by Dehn filling. Let $i: \partial Y \to Y$ be the inclusion map. Then it suffices to show that

(*) \quad \text{rank}[H_1(Y)/i_*(H_1(\partial Y))] \geq 1.
Given a finite index subgroup $G \subset \text{PSL}_2(\mathbb{Z}[\omega])$, denote by $U_G \subset G$ the (normal) subgroup generated by the parabolic matrices of $G$.

**Definition.** $d(G) = \dim_{\mathbb{Q}} ((G/U_G)^{\text{ab}} \otimes_{\mathbb{Z}} \mathbb{Q})$

**Lemma 3.** If $G' \subset G$ is of finite index, then $d(G') \geq d(G)$.

**Proof.** Since $G' \subset G$ is of finite index, the homomorphism, $(G'/U_{G'})^{\text{ab}} \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow (G/U_G)^{\text{ab}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is surjective. ♡

**Lemma 4.** $d(\Gamma \cap \Gamma(n)) = \text{rank}[H_1(Y)/i_*(H_1(\partial Y))]$

**Proof.** Follows from the isomorphism $\pi_1(Y) \cong \Gamma \cap \Gamma(n)$ and the correspondence between homotopy classes of loops in $\partial Y$ and parabolic matrices of $\Gamma \cap \Gamma(n)$. ♡

Now since $\Gamma \cap \Gamma(n) \subset \Gamma(n)$, (*) follows from Lemmas 3–4 and

**Lemma 5.** For $n \geq 5$, $d(\Gamma(n)) \geq 1$.

**Proof.** Let $\mathbb{Z}_n[\omega] \subset \mathbb{Z}[\omega]$ denote the order of index $n$. Since

$$\Gamma(n) \subset \text{PSL}_2(\mathbb{Z}_n[\omega])$$

is of finite index, it suffices to prove the lemma for $\text{PSL}_2(\mathbb{Z}_n[\omega])$, $n \geq 5$. Grunewald and Schwermer [GS] show that $d(\text{PSL}_2(\mathbb{Z}_n[\omega])) \geq \text{card}(W) - 1$, where $W$ is the set of natural numbers $m$ satisfying the following conditions:

(a) $m > 0$, $m \neq 2$, $(m, n) = 1$,
(b) $4m^2 \leq 3n^2 - 3$,
(c) every prime divisor of $m$ is inert in $\mathbb{Z}[\omega]$.

Thus if $n \geq 6$ and $(5, n) = 1$, then $\{1, 5\} \subset W$ and we are done. If $5|n$, then $\text{PSL}_2(\mathbb{Z}_n[\omega]) \subset \text{PSL}_2(\mathbb{Z}_5[\omega])$ and a computation gives $d(\text{PSL}_2(\mathbb{Z}_5[\omega])) \geq 1$. ♡

**References**


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