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THE VIRTUAL Z-REPRESENTABILITY OF CERTAIN 3-MANIFOLD GROUPS

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ABSTRACT. We use results on the cohomology of principal congruence subgroups of $\mathrm{PSL}_2(\mathbf{Z}[\omega])$, $\omega^2 + \omega + 1 = 0$, to prove the existence of a large class of closed, orientable 3-manifolds with virtually $\mathbf{Z}$-representable fundamental groups. In particular, these manifolds have finite covers with positive first Betti number.

1. Let M be a compact 3-manifold. We say that M has a virtually $\mathbf{Z}$-representable fundamental group if some finite index subgroup $G \subset \pi_1(M)$ maps epimorphically to $\mathbf{Z}$, or equivalently, if M has a finite sheeted cover $\widetilde{M}$ with rank $H_1(\widetilde{M}) \geq 1$. If M is compact, orientable, and irreducible, then the virtual $\mathbf{Z}$-representability of $\pi_1(M)$ implies that the above cover $\widetilde{M}$ is a Haken manifold (cf. [**H1**]). Waldhausen has conjectured that a closed, orientable, irreducible 3-manifold M with infinite fundamental group is virtually Haken i.e. is finitely covered by a Haken manifold. Hence a stronger version of this conjecture is that every such M has a virtually $\mathbf{Z}$-representable fundamental group.

In this paper we prove the existence of a large class of closed, orientable 3-manifolds with virtually $\mathbf{Z}$-representable fundamental groups. Let M be a closed, orientable 3-manifold. Then M can be realized as a branched cover of S^3, branched over the figure eight knot, K [**HLM**]. We prove:

THEOREM. *Let M be a closed, orientable 3-manifold which is a branched cover of S^3, branched over the figure eight knot with all branching indices divisible by a common integer $n \geq 5$. Then M has a virtually $\mathbf{Z}$-representable fundamental group.*

Note. Using different methods, Hempel [**H2**] has proved this result in the case where the branching indices are all equal to an odd integer $n \geq 3$.

In [**B**] we showed that for $n \geq 5$ the n-fold branched cyclic covers of S^3 branched over the figure eight knot had virtually $\mathbf{Z}$-representable fundamental groups. The above theorem generalizes this result.

2. Let $M \to S^3$ be branched over the figure eight knot, K, with all branching indices divisible by a common integer $n \geq 5$, and let $X \to N$ be the associated unbranched cover obtained by removing an open tubular neighborhood of the branch set K and its inverse image (thus ∂X, ∂N are disjoint unions of tori).

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We will use the fact that $\overset{\circ}{N} \cong S^3 \backslash K$ has an arithmetic hyperbolic structure. Specifically

$$\pi_1(N) = \langle x, y \mid (x^{-1}yxy^{-1})x(x^{-1}yxy^{-1})^{-1} = y \rangle$$

and there is a discrete, faithful representation $\pi_1(N) \to \mathrm{PSL}_2(\mathbf{Z}[\omega])$, where $\omega^2 + \omega + 1 = 0$, given by

$$x \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad y \mapsto \begin{bmatrix} 1 & 0 \\ -\omega & 1 \end{bmatrix}$$

(cf. [**R**]). Denote by Γ_K (resp. Γ) the image of $\pi_1(N)$ (resp. $\pi_1(X) \subset \pi_1(N)$) in $\mathrm{PSL}_2(\mathbf{Z}[\omega])$ under this representation.

A meridian loop μ in ∂N can be chosen so that $[\mu] = x$. In each component of ∂X there is a loop α_i that projects r_i to 1 onto μ, where r_i is the corresponding branching index in the branched cover $M \to S^3$. Hence it follows that the loop α_i corresponds to a homotopy class in $\pi_1(X)$ represented in Γ by an element of the form

$$R_i \begin{bmatrix} 1 & r_i \\ 0 & 1 \end{bmatrix} R_i^{-1},$$

where $R_i \in \Gamma_K$.

3. Consider the regular cover $Y \to X$ corresponding to the normal subgroup $\Gamma \cap \Gamma(n) \subset \Gamma$, where $\Gamma(n) \subset \mathrm{PSL}_2(\mathbf{Z}[\omega])$ is the n-principal congruence subgroup (cf. [**S**]).

LEMMA 1. *The loops α_i in ∂X lift to loops in the boundary tori of Y (hence these lifted loops project homeomorphically to the α_i).*

PROOF. The loops α_i in $\pi_1(X)$ are represented in Γ by elements of the form

$$R_i \begin{bmatrix} 1 & r_i \\ 0 & 1 \end{bmatrix} R_i^{-1}, \qquad R_i \in \Gamma_K,$$

which are also in $\Gamma \cap \Gamma(n)$ since

$$\begin{bmatrix} 1 & r_i \\ 0 & 1 \end{bmatrix} \in \Gamma(n)$$

($n \mid r_i$ by assumption) and $\Gamma(n)$ is normal in $\mathrm{PSL}_2(\mathbf{Z}[\omega])$. □

Now in each component of ∂Y covering the component of ∂X containing α_i, choose one lift β_{ij} of α_i. Then it follows that

LEMMA 2. *The cover $Y \to X$ extends to a regular (unbranched) cover $\widetilde{M} \to M$ by performing Dehn filling on X and on Y with respect to the loops $\{\alpha_i\}$ in ∂X and $\{\beta_{ij}\}$ in ∂Y.*

By Dehn filling on a 3-manifold P with respect to a loop in a boundary torus we mean attaching a solid torus to ∂P so that this loop bounds a meridional disk in the solid torus.

4. We complete the proof of our theorem by showing that the cover $\widetilde{M} \to M$ constructed in §3 satisfies rank $H_1(\widetilde{M}) \geq 1$. To see this, consider the cover $Y \to X$ from which we obtained $\widetilde{M} \to M$ by Dehn filling. Let $i \colon \partial Y \to Y$ be the inclusion map. Then it suffices to show that

$$(*) \qquad \mathrm{rank}[H_1(Y)/i_*(H_1(\partial Y))] \geq 1.$$

Given a finite index subgroup $G \subset \mathrm{PSL}_2(\mathbf{Z}[\omega])$, denote by $U_G \subset G$ the (normal) subgroup generated by the parabolic matrices of G.

DEFINITION. $d(G) = \dim_{\mathbf{Q}}((G/U_G)^{\mathrm{ab}} \otimes_{\mathbf{Z}} \mathbf{Q})$

LEMMA 3. *If $G' \subset G$ is of finite index, then $d(G') \geq d(G)$.*

PROOF. Since $G' \subset G$ is of finite index, the homomorphism, $(G'/U_{G'})^{\mathrm{ab}} \otimes_{\mathbf{Z}} \mathbf{Q} \to (G/U_G)^{\mathrm{ab}} \otimes_{\mathbf{Z}} \mathbf{Q}$ is surjective. □

LEMMA 4. $d(\Gamma \cap \Gamma(n)) = \mathrm{rank}[H_1(Y)/i_*(H_1(\partial Y))]$

PROOF. Follows from the isomorphism $\pi_1(Y) \cong \Gamma \cap \Gamma(n)$ and the correspondence between homotopy classes of loops in ∂Y and parabolic matrices of $\Gamma \cap \Gamma(n)$. □

Now since $\Gamma \cap \Gamma(n) \subset \Gamma(n)$, $(*)$ follows from Lemmas 3–4 and

LEMMA 5. *For $n \geq 5$, $d(\Gamma(n)) \geq 1$.*

PROOF. Let $\mathbf{Z}_n[\omega] \subset \mathbf{Z}[\omega]$ denote the order of index n. Since

$$\Gamma(n) \subset \mathrm{PSL}_2(\mathbf{Z}_n[\omega])$$

is of finite index, it suffices to prove the lemma for $\mathrm{PSL}_2(\mathbf{Z}_n[\omega])$, $n \geq 5$. Grunewald and Schwermer [**GS**] show that $d(\mathrm{PSL}_2(\mathbf{Z}_n[\omega])) \geq \mathrm{card}(W) - 1$, where W is the set of natural numbers m satisfying the following conditions:

(a) $m > 0$, $m \neq 2$, $(m, n) = 1$,

(b) $4m^2 \leq 3n^2 - 3$,

(c) every prime divisor of m is inert in $\mathbf{Z}[\omega]$.

Thus if $n \geq 6$ and $(5, n) = 1$, then $\{1, 5\} \subset W$ and we are done. If $5|n$, then $\mathrm{PSL}_2(\mathbf{Z}_n[\omega]) \subset \mathrm{PSL}_2(\mathbf{Z}_5[\omega])$ and a computation gives $d(\mathrm{PSL}_2(\mathbf{Z}_5[\omega])) \geq 1$. □

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