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NONTANGENTIAL MAXIMAL FUNCTIONS OVER COMPACT RIEMANNIAN MANIFOLDS

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ABSTRACT. The nontangential maximal function associated to the Poisson semigroup for a compact Riemannian manifold is shown to be weak type (1,1) and L^p bounded ($p > 1$).

Let M be a compact Riemannian manifold with Riemannian measure dV and Laplace-Beltrami operator Δ. Let $p_t(x,y)$ be the fundamental solution to $L = \partial^2/\partial t^2 + \Delta = 0$; thus, for f continuous on M, $P_t f(y) = \int_M p_t(x,y) f(x)\,dV(x)$ satisfies $LP_t f = 0$ and $\lim_{t\to 0} P_t f(y) = f(y)$ for each y in M. It is a standard fact that when the class of boundary functions is suitably enlarged, almost everywhere convergence still holds. Indeed, a stronger theorem, proved by C. S. Herz in a more general context [**3**], is that the radial maximal function $P_* f = \sup_{t>0} P_t|f|$ is bounded on $L^p(M)$ for $p > 1$ and is of weak type (1,1) (cf. [**4**, p. 48] for the latter case). Let $d(x,y)$ be the distance function on M associated to the Riemannian metric, let $\Gamma_\gamma(y) = \{(y',t) \in M \times (0,\infty) \mid d(y',y) \leq \gamma t\}$ for each positive γ, and let $N_\gamma f$ be the nontangential maximal function defined by

$$N_\gamma f(y_0) = \sup_{(y,t)\in\Gamma_\gamma(y_0)} \int_M p_t(x,y)|f(x)|\,dV(x).$$

In this note we show that N_γ is $L^p(M)$ bounded ($p > 1$) and of weak type (1,1). Standard technique then allows the conclusion of nontangential boundary convergence

$$\lim_{\substack{t\to 0\\ (y,t)\in\Gamma_\gamma(y_0)}} P_t f(y) = f(y_0)\,dV\text{-a.e.} \quad \text{for } f \text{ in } L^p(M)\ (p \geq 1),$$

although this result is otherwise easily obtainable by P.D.E. methods. Our method is to prove a two-sided inequality for the Poisson kernel $p_t(x,y)$ which immediately reduces our theorem to Herz's. This inequality is obtained by utilizing known behavior of the heat kernel in conjunction with the subordination of the Poisson semigroup to the heat diffusion semigroup, a special case of Bochner's principle of subordination. The author used these ideas in an earlier paper where M was either a compact semisimple Lie group (superseded by the current paper) or a symmetric space of noncompact type [**1**]; in both cases exact formulae for the heat kernel were used whereas in the present paper we use the Minakshisundaram-Pleijel asymptotic expansion.

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Let $k_t(x,y)$ be the fundamental solution to the heat diffusion operator $\partial/\partial t - \Delta$. We require two basic facts concerning the asymptotics of $k_t(x,y)$ as $t \to 0$ and as $t \to \infty$ (cf. [**2**, Chapter 6]):

$$\lim_{t\to\infty} k_t(x,y) = 1/\operatorname{Vol} M \text{ uniformly on } M \times M. \tag{1}$$

$$k_t(x,y) = (4\pi t)^{-n/2} e^{-d^2(x,y)/4t} \left\{ \sum_{k=0}^{n+2} t^k w_k(x,y) + O(t^{n+3}) \right\} \tag{2}$$

as $t \to 0^+$. The $w_k(x,y)$ are differentiable functions and $n = \dim M$.

The first formula is an easy consequence of the uniform dissipation of heat principle [**2**, p. 141]; the second is a version of the classical Minakshisundaram-Pleijel expansion [**2**, p. 154]. To avoid carrying the constant in (1) we assume without loss of generality that the volume of M is 1. We also require the simple consequence of selfadjointness of the heat diffusion semigroup together with the maximum principle that

$$k_t(x,y) > 0. \tag{3}$$

All of these properties can be suitably transferred to the Poisson kernel by means of the subordination formula

$$p_t(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} k_{t^2/4u}(x,y)\, du \tag{4}$$

or the equivalent

$$p_t(x,y) = \frac{t}{2\sqrt{\pi}} \int_0^\infty v^{-3/2} e^{-t^2/4v} k_v(x,y)\, dv. \tag{5}$$

LEMMA. *The Poisson kernel $p_t(x,y)$ is strictly positive and has limit* 1 *uniformly as $t \to \infty$.*

PROOF. The first statement is a trivial consequence of (3) and (4). Given $\varepsilon > 0$ choose δ so that $k_v(x,y) \in (1-\varepsilon, 1+\varepsilon)$ for $v > \delta$. By (2) $k_v(x,y) \le c(\delta) v^{-n/2}$ for $v < \delta$. Dividing the integral in (5) into segments $(0,\delta)$ and (δ,∞), the dominated convergence theorem shows that

$$\lim_{t\to\infty} t \int_0^\delta v^{-(n+3)/2} e^{-t^2/4v}\, dv = 0,$$

and since ε is arbitrary, it suffices to show that

$$\lim_{t\to 0} t \int_\delta^\infty v^{-3/2} e^{-t^2/4v}\, dv = 2\sqrt{\pi}.$$

This is obvious after the change of variables $u = t^2/4v$. $\square$

PROPOSITION. *For each $\gamma > 0$ there exists a constant $C = C_{\gamma,M}$ depending only on γ and M such that*

$$p_t(x,y') \le C p_t(x,y) \qquad (x,y \in M\colon (y',t) \in \Gamma_\gamma(y)). \tag{6}$$

PROOF. First it is clear from the preceding lemma and from (1) and (3) that if ϕ_t is either p_t or k_t then $\sup_{t>\delta}\{\phi_t(x_1,y_1)/\phi_t(x_2,y_2) | x_1,x_2,y_1,y_2 \in M\}$ is finite for every $\delta > 0$; in particular, (6) is at issue only for small t. Similarly we may choose

δ so that the brace-bracketed term in (2) is bounded between $\alpha = \frac{1}{2}\min\{w_0(x,y) \mid x,y \in M\}$ and $\beta = 2\max\{w_0(x,y) \mid x,y \in M\}$ for $t < \delta$. Thus, for δ suitably small there are positive constants α, β, and Ω such that for every x, y and y' in M we have

$$(7)\quad \alpha(4\pi v)^{-n/2}e^{-d^2(x,y)/4v} \le k_v(x,y) \le \beta(4\pi v)^{-n/2}e^{-d^2(x,y)/4v} \qquad (0 < v < 4\delta)$$

and

$$(8)\qquad k_v(x,y') \le \Omega k_v(x,y) \qquad (\delta < v < \infty).$$

Fix x and y in M, $t > 0$, and y' with $d(y',y) \le \gamma t$. We divide the proof into two cases. To begin, we suppose that $d(x,y) \ge 2\gamma t$. It follows from the triangle inequality that $d(x,y) \le 2d(x,y')$. Divide the integral formula (4) for $p_t(x,y')$ into the two segments $(0, t^2/4\delta)$ and $(t^2/4\delta, \infty)$ where δ is as above:

$$p_t(x,y') = \pi^{-1/2}\left(\int_0^{t^2/4\delta} + \int_{t^2/4\delta}^{\infty}\right) u^{-1/2}e^{-u}k_v(x,y')\,du$$

where $v = t^2/4u$. In the first integral, that is where $v > \delta$, we use (8) and then increase the upper limit of the first integral to conclude

$$p_t(x,y') < \Omega p_t(x,y) + \pi^{-1/2}\int_{t^2/4\delta}^{\infty} u^{-1/2}e^{-u}k_v(x,y')\,du.$$

In the remaining integral we may use (7); since both v and $4v$ are smaller than 4δ we have

$$\begin{aligned} k_v(x,y') &\le \beta(4\pi v)^{-n/2}e^{-d^2(x,y')/4v} \\ &\le \beta(4\pi v)^{-n/2}e^{-d^2(x,y)/16v} \le 2^n(\beta/\alpha)k_{4v}(x,y). \end{aligned}$$

Thus

$$p_t(x,y') < \Omega p_t(x,y) + 2^n\left(\frac{\beta}{\alpha}\right)\pi^{-1/2}\int_0^{\infty} u^{-1/2}e^{-u}k_{t^2/u}(x,y)\,du.$$

On making the change of variable $u = 4w$ and the trivial comparison $e^{-4w} < e^{-w}$ we get

$$p_t(x,y') < (\Omega + 2^{n+1}\beta/\alpha)p_t(x,y).$$

In the second case we have $d(x,y) \le 2\gamma t$. We may choose a constant C_1 sufficiently large so that

$$(9)\qquad \int_{\tau}^{\infty} u^{(n-1)/2}e^{-u}\,du \le C_1\int_{\tau}^{\infty} u^{(n-1)/2}e^{-u}e^{-4\gamma^2 u}\,du$$

for each τ in $[0,\delta]$. As remarked earlier, (6) is obvious for t larger than any given fixed lower bound; we restrict t to $(0, 2\delta)$, then for $0 < t < 2\delta$ and $d(x,y) \le 2\gamma t$ we have

$$(10)\qquad \int_{t^2/4\delta}^{\infty} u^{(n-1)/2}e^{-u}\,du \le C_1\int_{t^2/4\delta}^{\infty} u^{(n-1)/2}e^{-u}e^{-ud^2(x,y)/t^2}\,du.$$

Splitting the integral (4) for $p_t(x, y')$ as in the first case we have, in view of (7), (8), and (10),

$$\begin{aligned}
p_t(x, y') &< \Omega p_t(x, y) + \pi^{-1/2} \int_{t^2/4\delta}^{\infty} u^{-1/2} e^{-u} k_v(x, y')\, du \\
&< \Omega p_t(x, y) + \beta(\pi t^2)^{-n/2} \pi^{-1/2} \int_{t^2/4\delta}^{\infty} u^{(n-1)/2} e^{-u}\, du \\
&< \Omega p_t(x, y) + \beta C_1 (\pi t^2)^{-n/2} \pi^{-1/2} \int_{t^2/4\delta}^{\infty} u^{(n-1)/2} e^{-u} e^{-ud^2(x,y)/t^2}\, du \\
&< \Omega p_t(x, y) + \frac{\beta C_1}{\alpha} \int_{t^2/4\delta}^{\infty} u^{-1/2} e^{-u} k_{t^2/4u}(x, y)\, du \\
&< (\Omega + \beta C_1/\alpha) p_t(x, y). \quad \square
\end{aligned}$$

It follows immediately from this proposition that $P_* f \leq N_\gamma f \leq C_{\gamma,M} \cdot P_* f$ ($f \in L^p(M)$, $1 \leq p \leq \infty$) pointwise. Thus,

THEOREM. *There is a constant A depending only on p, M and γ such that for f in $L^p(M)$, $1 \leq p \leq \infty$,*

$$\|N_\gamma f\|_{L^p(M)} \leq A \|f\|_{L^p(M)} \qquad (1 < p \leq \infty)$$

and for all $\lambda > 0$

$$dV\{|f| > \lambda\} \leq (A/\lambda) \|f\|_{L^1(M)} \qquad (p = 1).$$

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