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JOINT CONTINUITY OF MEASURABLE BIADDITIVE MAPPINGS

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ABSTRACT. The main result of this paper is the following theorem. If G_1, G_2 and G_3 are abelian Polish groups and $C: G_1 \times G_2 \to G_3$ is a Christensen measurable biadditive mapping, then C is jointly continuous.

1. Introduction. Biadditive mappings have been discussed many times in the literature, often in conjunction with the parallelogram law [**10**, **8**, **5**, **7**], etc. An important problem in the theory of biadditive mappings is to find conditions which ensure the continuity of the mappings.

There are many results about continuity of measurable homomorphisms in quite general settings but in the case of biadditive mappings we are aware only of studies of measurable solutions of the associated parallelogram law in a general (Banach space) setting. [**9**]

In this work it will be assumed that the domain of the biadditive mappings is the product of two abelian Polish (or more general) groups and the range space is again an abelian Polish (or more general) group. Hence, in general, there is no associated parallelogram law with the aid of the diagonal of the domain; thus only the structure of the group can be used.

We wil discuss separately continuous and measurable biadditive mappings, where both universal and Christensen measurability will be considered. We refer to [**2**] for the definition of the Haar zero sets in an abelian Polish group and to [**6** and **4**] for the notion of Christensen measurability. Since there are no Fubini types of theorem for the small sets we consider [**2**], a quite intricate construction has to be given to prove our main result. Nevertheless, we believe that these results show the power of these small sets.

Throughout this paper we shall use the following notations. If G_i is an abelian Polish group, then O_{G_i} is the neutral element of G_i, and d_i is a complete translation invariant metric, which is compatible with the topology of G_i.

2. Universally measurable mappings. Our first result shows that a universally measurable biadditive mapping is jointly continuous in an abelian Polish group setting.

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THEOREM 1. *Let G_1, G_2 and G_3 be abelian Polish groups and let $C\colon G_1 \times G_2 \to G_3$ be a universally measurable biadditive mapping. Then C is jointly continuous.*

PROOF. Let g_1 be a fixed element of G_1, then $C(g_1, \cdot)\colon G_2 \to G_3$ is a group homomorphism. In an obvious way, G_2 can be identified with the closed subset $\{g_1\} \times G_2$ of $G_1 \times G_2$ (the mapping $g_2 \mapsto g_1 \times g_2$ is a homomorphism). Now, using the fact that a closed set is also universally measurable, and that the restriction of a universally measurable mapping to a closed subset is relatively universally measurable, we can conclude that $C(g_1, \cdot)$ is a universally measurable group homomorphism. Therefore $C(g_1, \cdot)$ is continuous [**3**, p. 114]. In an entirely similar way, we can show that $C(\cdot, g_2)\colon G_1 \to G_3$ is a continuous group homomorphism for every fixed $g_2 \in G_2$. Hence $C\colon G_1 \times G_2 \to G_3$ is a separately continuous mapping. With the aid of a result of Calbrix and Troallic [**1**, Corollary 3, p. 648] we can conclude that there exists at least one $g_1 \in G_1$ and at least one $h_\infty \in G_2$ so that (g_1, h_∞) is a point of joint continuity for C. To conclude the proof, we shall show that all the points of $G_1 \times G_2$ are points of joint continuity for C.

Indeed, let $\{g'_n\}$ be a sequence from G_1 such that $\lim_{n\to\infty} g'_n = O_{G_1}$, and let $\{h_n\}$ be a sequence from G_2 such that $\lim_{n\to\infty} h_n = h_\infty$. Since (g_1, h_∞) is a point of joint continuity for C, we have that $\lim_{n\to\infty} C(g_1 + g'_n, h_n) = C(g_1, h_\infty)$. Now, the biadditivity of C implies that $C(g_1 + g'_n, h_n) = C(g_1, h_n) + C(g'_n, h_n)$. Since $C(g_1, \cdot)\colon G_2 \to G_3$ is continuous, we see that

$$\lim_{n\to\infty} C(g_1 + g'_n, h_n) = \lim_{n\to\infty} C(g_1, h_n).$$

Therefore $\lim_{n\to\infty} C(g'_n, h_n) = O_{G_3}$, and clearly this last result remains valid for any other sequences such that $g^1_n \to O_{G_1}$ and $h^1_n \to h_\infty$. Now, let $\{g^*_n\}$ be a sequence from G_1 such that $\lim_{n\to\infty} g^*_n = g_1$ and let $\{h^*_n\}$ be a sequence from G_2 such that $\lim_{n\to\infty} h^*_n = O_{G_2}$. Then, a similar argument shows that $\lim_{n\to\infty} C(g^*_n, h^*_n) = O_{G_3}$. From which, using the biadditivity of C, we can deduce easily that C is jointly continuous at every point of $G_1 \times G_2$.

The second half of the proof of Theorem 1 together with [**1**, Corollary 4, p. 648] yield the following result.

THEOREM 2. *Let G_1, G_2 and G_3 be abelian complete metrizable groups. Let $C\colon G_1 \times G_2 \to G_3$ be biadditive and separately continuous. Then C is jointly continuous.*

3. Christensen measurable mappings. First we prove the following lemma.

LEMMA 1. *Let G be an abelian Polish group. Let S be a universally measurable subset of G such that $G \setminus S = S^c$ is a Haar zero set. Let u be a probability measure on G. Then there exists a $g \in G$ such that $u(S + g) = 1$.*

PROOF. Let v be a probability 'testing' measure for S^c, i.e.

$$(\chi_{S^c} * v)(x) = \int_G \chi_{S^c}(x + y) v(dy) = 0.$$

Then by the Fubini theorem, we have that

$$(u * v)(S^c) = \iint \chi_{S^c}(x + y)\, du(x)\, dv(y) = \iint \chi_{S^c}(x + y)\, dv(y)\, du(x) = 0.$$

Hence $u(S^c + g) = 0$ for v a.e. g. Therefore for v a.e. g we have that $u(S + g) = 1$.

Now, we can prove our main result.

THEOREM 3. *Let G_1, G_2 and G_3 be abelian Polish groups, let $C\colon G_1 \times G_2 \to G_3$ be a Christensen measurable biadditive mapping. Then C is jointly continuous.*

PROOF. As the first step, we show that there is a universally measurable set $S^c G_1 \times G_2$ such that the restriction of C to S is universally measurable and $S^c = (G_1 \times G_2) \setminus S$ is a Haar null set. To see that, let $\{O_n\}$ be a countable base for the topology of G_3. Since $C^{-1}(O_n)$ is Christensen measurable, we have that $C^{-1}(O_n) = A_n \cup B_n$, where A_n is universally measurable and B_n is a Christensen zero set. Hence there exists an N_n which is a Haar null set and so that $B_n \subset N_n$. Furthermore, $S_n = C^{-1}(O_n) \setminus N_n$ is universally measurable. Let $S = \bigcup_{n=1}^{\infty} S_n$, it is easy to show that S has the required properties.

As the second step, we show that C is separately continuous. To obtain a contradiction, suppose $g_0 \in G_1$ is such that $C(g_0, \cdot)\colon G_2 \to G_3$ is discontinuous. Then there exist a sequence $\{h_n\}$ and $\varepsilon_0 > 0$ so that, $h_n \in G_2$, $d_2(O_{G_2}, h_n) < 1/2^n$, and $d_3(O_{G_3}, C(g_0, h_n)) \geq \varepsilon_0$.

Let $H = \{0,1\}$ be the compact metrizable abelian group with Haar measure μ_1 such that $\mu_1(\{0\}) = \mu_1(\{1\}) = \frac{1}{2}$. We define $\theta_1\colon H \to G_1$ by $\theta_1(x) = x g_0$. Let $K = \{0,1\}^N$ be the Cantor group with Haar measure μ_2 which is the (countable) product of 'equidistribution' measures on $\{0,1\}$. We define $\theta_2\colon K \to G_2$ by $\theta_2(y) = \sum_{n=1}^{\infty} y(n) h_n$ and $\theta\colon H \times K \to G_1 \times G_2$ by $\theta(x,y) = (\theta_1(x), \theta_2(y))$. Clearly, θ_1, θ_2 and θ are continuous mappings. By Lemma 1 we conclude that there exists a $(g_1, g_2) \in G_1 \times G_2$ so that $\theta(H \times K) + (g_1, g_2)$ is essentially contained in S, in other words, $\theta^{-1}(S - (g_1, g_2))$ has full Haar measure in $H \times K$. Therefore $D(x,y) = C(\theta(x,y) + (g_1, g_2))$ is Haar measurable on $H \times K$, since it follows from our previous result that the composed map D will be universally measurable on $H \times K$, except for a null set.

Let ε be a fixed positive number. There exist balls B_0 and B_1 in G_3 of diameter less than ε so that both of the sets $F_0 = (\{0\} \times K) \cap D^{-1}(B_0)$ and $F_1 = (\{1\} \times K) \cap D^{-1}(B_1)$ have strictly positive Haar measure in $H \times K$. Now, by Pettis' theorem, we can conclude that there exists $N(\varepsilon)$ so that for $n \geq N(\varepsilon)$ we have that $(F_0 + (\{0\} \times l_n)) \cap F_0 \neq \varnothing$ and $(F_1 + (\{0\} \times l_n)) \cap F_1 \neq \varnothing$, where $l_n = (0, 0, \ldots, 0, 1, 0, \ldots)$.

Since $F_0 = (\{0\} \times K) \cap \{(0,g)\colon C(g_1, \theta_2(g) + g_2) \in B_0\}$ and $F_1 = (\{1\} \times K) \cap \{(1,g)\colon C(g_0 + g_1, \theta_2(g) + g_2) \in B_1\}$, it follows that for $n \geq N(\varepsilon)$ $d_3(O_{G_3}, C(g_1, h_n)) < \varepsilon$ and $d_3(O_{G_3}, C(g_0 + g_1, h_n)) < \varepsilon$.

Now these inequalities imply that $d_3(O_{G_3}, C(g_0, h_n)) < 2\varepsilon$. Thus, by choosing $\varepsilon < \frac{1}{2}\varepsilon_0$, we got a contradiction, i.e. we have that C is separately continuous.

Finally, we see that Theorem 1 yields the proof of this theorem.

REFERENCES

1. J. Calbrix and J. P. Troallic, *Applications séparément continues*, C. R. Acad. Sci. Paris **288** (1979), 647–648.
2. J. P. R. Christensen, *On sets of Haar measure zero in abelian Polish groups*, Israel J. Math. **13** (1972), 255–260.
3. ———, *Topology and Borel structure*, North-Holland, Amsterdam, American Elsevier, New York, 1974.
4. J. P. R. Christensen and P. Fischer, *Small sets and a class of general functional equations*, Aequationes Math. (to appear).

5. T. M. K. Davison, *A functional equation for quadratic forms*, Aequationes Math. **21** (1980), 1–7.
6. P. Fischer and Z. Slodkowski, *Christensen zero sets and measurable convex functions*, Proc. Amer. Math. Soc. **79** (1980), 449–453.
7. P. Fischer and J. P. Mokanski, *A class of symmetric biadditive functionals*, Aequationes Math. **23** (1981), 189–174.
8. A. M. Gleason, *The definition of a quadratic form*, Amer. Math. Monthly **73** (1966), 1049–1056.
9. S. Kurepa, *On the quadratic functional*, Acad. Serbe. Sci. Publ. Inst. Math. **13** (1959), 57–73.
10. P. Jordan and J. von Neumann, *On inner products in linear metric spaces*, Ann. of Math. (2) **36** (1935), 719–723.

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