

## ELEMENTARY DIVISOR THEOREM FOR NONCOMMUTATIVE PIDS

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**ABSTRACT.** We prove that, over a PID, if two matrices  $\mathbf{A}$  and  $\mathbf{B}$  have the same size, present isomorphic modules and have rank  $\geq 2$ , then  $\mathbf{A}$  is equivalent to  $\mathbf{B}$ . This answers a question raised by Nakayama in 1938. Our solution makes use of a number of facts about the algebraic  $K$ -theory of noetherian rings.

Let  $\Lambda$  be a PID, that is, an integral domain (not necessarily commutative) in which every left ideal and every right ideal is principal. We call  $m \times n$  matrices  $\mathbf{A}$  and  $\mathbf{B}$  over  $\Lambda$  *equivalent*, and write  $\mathbf{A} \sim \mathbf{B}$ , if  $\mathbf{B} = \mathbf{P}\mathbf{A}\mathbf{Q}$  for invertible matrices  $\mathbf{P}$  and  $\mathbf{Q}$  over  $\Lambda$ . It is well known [T '37, A '38, J '43] that every such  $\mathbf{A}$  is equivalent to a diagonal matrix  $\mathbf{D}$  in which each diagonal entry is a *total divisor* of the next, that is,

$$(0.1) \quad \mathbf{D} = \text{diag}(d_1, d_2, \dots) \quad \text{where each } \Lambda d_i \cap d_i \Lambda \supseteq \Lambda d_{i+1} \Lambda.$$

Three obvious invariants for the equivalence class of  $\mathbf{A}$  are  $m, n$  and the isomorphism class of the left  $\Lambda$ -module

$$(0.2) \quad U = \Lambda^n / (\Lambda^m \mathbf{A}) \cong \bigoplus_i \Lambda / \Lambda d_i$$

presented by  $\mathbf{A}$ . Nakayama [N '38] refined this by observing that the isomorphism classes of the left  $\Lambda$ -modules  $\Lambda / \Lambda d_i$  (each counted as often as it occurs) are invariants of the equivalence class of  $\mathbf{A}$ . However, the main purpose of his paper was to lament the fact that very little is known about the converse question: What invariants other than  $m, n$  and the isomorphism class of  $U$  are needed to determine the equivalence class of  $\mathbf{A}$ ?

It was apparently well known that additional invariants are needed, even in the case of  $1 \times 1$  matrices. For explicit examples, see [LR '74, 4.6, §5] and [GL '88]. The surprising answer to Nakayama's question is that, except in rank 1, there are no additional invariants:

**ELEMENTARY DIVISOR THEOREM.** *If rank( $\mathbf{A}$ )  $\geq 2$ , then  $m, n$  and the isomorphism class of  $U$  form a complete set of invariants for the equivalence class of  $\mathbf{A}$ .*

For completeness we note that the situation where rank( $\mathbf{A}$ ) = 1 but  $(m, n) \neq (1, 1)$  is essentially the same as the  $1 \times 1$  case. Let  $[a]$  and  $[b]$  be inequivalent  $1 \times 1$

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matrices such that  $\Lambda/\Lambda a \cong \Lambda/\Lambda b$ , and let  $\mathbf{A}$  and  $\mathbf{B}$  be  $m \times n$  matrices with  $a$  and  $b$  respectively, in their  $(1, 1)$ -position and zeros elsewhere. Then  $\mathbf{A}$  and  $\mathbf{B}$  are inequivalent but the  $\Lambda$ -module (0.2) presented by  $\mathbf{A}$  is isomorphic to the analogous  $\Lambda$ -module presented by  $\mathbf{B}$ .

The Elementary Divisor Theorem above was proved in [GL '88] for the case that  $\Lambda$  is module-finite over its center. In the present paper, we show how to extend that proof to the full noncommutative situation.

OUTLINE OF PROOF. It is convenient to take a more abstract point-of-view, replacing matrices with module homomorphisms. We call  $\Lambda$ -module homomorphisms  $f, g: M \rightarrow N$  (often acting on the right) *equivalent*, and write  $f \sim g$ , if  $g = \varphi f \vartheta$  for automorphisms  $\varphi$  and  $\vartheta$  of  $M$  and  $N$  respectively. Letting  $\alpha$  denote right multiplication by the  $m \times n$  matrix  $\mathbf{A}$  yields a free resolution

$$(0.3) \quad \Lambda^m \xrightarrow{\alpha} \Lambda^n \xrightarrow{f} U$$

where  $U$  is the module in (0.2). Similarly, right multiplication by  $\mathbf{B}$  yields another resolution  $(\beta, g)$  of  $U$ , since we are assuming that  $\mathbf{A}$  and  $\mathbf{B}$  present isomorphic modules.

We want to prove that  $\alpha \sim \beta$ .

The problem is easily reduced to the case that  $\alpha$  is one-to-one, as in the proof of [GL '88, 3.6]. (The kernel of  $\alpha$  splits off since  $\Lambda$  is a PID.) Thus, from now on we will be concerned with a presentation

$$(0.4) \quad K \xrightarrow{\alpha} P = \Lambda^n \xrightarrow{f} U$$

of a left  $\Lambda$ -module  $U$ , where  $\alpha$  denotes the inclusion map. What we want to prove is that  $U$  is *uniquely presentable* by  $P$ , that is, if  $g: P \rightarrow U$  is any other presentation of  $U$  by  $P$ , then  $g \sim f$ .

It is easy to reduce the problem to the case that  $U$  has finite length, as in the proof of [GL '88, 3.5] (because the torsion submodule of  $U$  splits off).

The first part of what we prove in this paper is that the theorem is "stably" true, that is, there is an  $s$  such that  $U^s$  is uniquely presentable by  $P^s$ . This is done in §1.

It follows that  $f^s \sim (g \oplus f^{s-1})$  since both are presentations of  $U^s$  by  $P^s$ . The final step is then to show that, when the free module  $P$  has rank  $\geq 2$ ,  $f^{s-1}$  can be cancelled, yielding  $f \sim g$ . This is done in §§2 and 3.

There are two difficulties in this proof that are not present when  $\Lambda$  is module-finite over its center. First is the presence of *completely faithful*  $\Lambda$ -modules of finite length, that is, faithful modules  $U$  of finite length such that every nonzero submodule of every homomorphic image of  $U$  is faithful. We reduce the cancellation problem to the case of presentations of unfaithful modules in §2, and then do the unfaithful case in §3. It is in this last part that the ideas from  $K$ -theory are used.

The second difficulty is that localization at maximal ideals of the center of  $\Lambda$  (one of the main tools in [GL '88]) does not seem very useful in the present situation, where  $\Lambda$  may not be module-finite over its center and we do not know very much about the center itself. Fortunately, we are able to avoid localization, in the present paper.

For the remainder of this paper,  $\Lambda$  denotes a PID, and “module” means “finitely generated module” unless otherwise stated. We want to show that every left  $\Lambda$ -module  $U$  of finite length is uniquely presentable by every free module  $P$  by which it can be presented, as in (0.4).

**1. Stable unique presentability.** We begin by quoting [LR '74, 1.5].

**1.1 LIFTING AND STRAIGHTENING THEOREM.** *Consider a surjection of modules over any ring*

$$(1.1.1) \quad f: P = P_1 \oplus P_2 \oplus \cdots \oplus P_n \rightarrow U = U_1 \oplus U_2 \oplus \cdots \oplus U_n$$

where  $P$  is projective and  $U/\text{rad}U$  is semisimple (e.g. any module  $U$  of finite length). Suppose that

$$(1.1.2) \quad \text{For each } i \text{ there exist surjections: } P_i \rightarrow U_i \text{ and } P_n \rightarrow U_i.$$

Then the  $P_i$  can be isomorphically replaced in (1.1.1) to achieve  $f(P_i) = U_i$  for every  $i$  (“lifting”). Moreover, this replacement can be done in such a way that the first  $n - 1$  restricted maps  $f: P_i \rightarrow U_i$  become equivalent to  $n - 1$  arbitrarily selected surjections:  $P_i \rightarrow U_i$  (“straightening”).

The phrase “isomorphically replaced”, in the theorem, means that the new  $P_i$  is isomorphic to the old one. An immediate consequence of the theorem is

**1.2 COROLLARY.** *Suppose that  $U_n$  is uniquely presentable by  $P_n$ . Then  $U$  is uniquely presentable by  $P$ .*

One situation in which unique presentability is easily shown to hold is given in [LR '74, 1.9].

**1.3 LEMMA.** *Let  $T$  be a 2-sided ideal of any ring  $R$ , and let  $H$  be any noetherian  $R$ -module. Then any two surjections:  $H \rightarrow H/TH$  are equivalent.*

We now return to our PID  $\Lambda$ .

**1.4 STABLE INVARIANT FACTOR THEOREM.** *Let  $K$  be a submodule of a free  $\Lambda$ -module  $P$ . Suppose that  $P/K$  is unfaithful of finite length. Then for some  $s$  there exist compatible decompositions*

$$(1.3.1) \quad P^s = P_1 \oplus \cdots \oplus P_n \text{ and}$$

$$(1.3.2) \quad K^s = T_1 P_1 \oplus \cdots \oplus T_n P_n \quad (T_1 \subseteq \cdots \subseteq T_n)$$

with each  $T_i$  a 2-sided ideal of  $\Lambda$  and each  $P_i \cong \Lambda$ .

**PROOF.** The proof is the same as the proof of [GL '88, 3.3], with the following minor changes. Since  $\Lambda$  is a PID, all projective  $\Lambda$ -modules are free, and their uniform rank equals their rank as a free module. Moreover, the “genus” of a projective module merely becomes its isomorphism class. In a number of places, the preliminary results [GL '88, 3.1, 3.2] refer to a regular element  $d$  of a central subring  $R$  such that  $dU = 0$ . In all such cases, it suffices to take  $d$  to be a regular element of  $R$ ; and this always exists, by Goldie’s theorem, because the annihilator of the unfaithful module  $U$  is a 2-sided ideal of the prime ring  $\Lambda$ , hence is essential as a left ideal.  $\square$

We now obtain our desired stable unique presentability result.

1.5 THEOREM. *Let  $f: P \rightarrow U$  be a presentation of a  $\Lambda$ -module of finite length. Then, for some  $s$ ,  $U^s$  is uniquely presentable by  $P^s$ .*

PROOF. We seek an integer  $s$  for which there exist decompositions  $P^s = \bigoplus_{i=1}^n P_i$  and  $U^s = \bigoplus_{i=1}^n U_i$  with each  $P_i \cong \Lambda$ , each  $U_i$  cyclic, and  $U_n \cong \Lambda/T$  for some 2-sided ideal  $T$  of  $\Lambda$ . For then  $U_n$  is uniquely presentable by  $P_n$  (Lemma 1.3), and hence  $U$  is uniquely presentable by  $P$  (Corollary 1.2).

Let  $K = \ker(f)$ . It suffices to find an integer  $s$  and decompositions

$$(1.5.1) \quad P^s = \bigoplus_{i=1}^n \Lambda p_i \quad \text{and} \quad K^s = \bigoplus_{i=1}^n \Lambda d_i p_i$$

such that  $T = \Lambda d_n$  is a 2-sided ideal, for then  $U = \bigoplus_i f(\Lambda p_i)$  with  $f(\Lambda p_n) \cong \Lambda/T$ .

We can suppose that  $P$  has rank  $r \geq 2$ , after replacing  $f$  by  $f^2$ , if necessary. Let  $K = \ker(f)$ . Applying the diagonalization theorem mentioned in (0.1) to the matrix of the inclusion map:  $K \subseteq P$ , we get a pair of decompositions

$$(1.5.2) \quad P = \bigoplus_{i=1}^r \Lambda p_i \quad \text{and} \quad K = \bigoplus_{i=1}^r \Lambda d_i p_i$$

where each  $d_i$  is a total divisor of  $d_{i-1}$  ( $i > 1$ ). Since  $U$  has finite length and  $P$  rank( $P$ ) =  $r$ , every  $d_i$  and  $p_i$  is nonzero.

Let  $P' = \bigoplus_{i \neq 1} \Lambda p_i$  and  $K' = \bigoplus_{i \neq 1} \Lambda d_i p_i$ . Since every  $d_i$  is a total divisor of  $d_{i-1}$  we have  $\Lambda d_1 \Lambda \subseteq \Lambda d_i$  for all  $i > 1$ . Therefore  $P'/K'$  is annihilated by  $d_1$ . Let  $s$  be the integer obtained by applying the Stable Invariant Factor Theorem to the unfaithful module  $P'/K'$ .

Since  $P = \Lambda p_1 \oplus P'$  and  $K = \Lambda d_1 p_1 \oplus K'$ , the decompositions of  $(P')^s$  and  $(K')^s$  provided by the Stable Invariant Factor Theorem yield the decomposition needed in (1.5.1).  $\square$

**2. Reduction to unfaithful modules.**

2.1 THEOREM. *Suppose that every unfaithful  $\Lambda$ -module  $U$  of finite length is uniquely presentable by every free module of rank  $\geq 2$  that can be mapped onto  $U$ . Then the same conclusion holds if  $U$  is faithful (and of finite length).*

PROOF. Let  $f: P \rightarrow U$  be a presentation with  $\ker(f) = K$ , where  $U$  is not necessarily faithful. Choose a pair of decompositions, as in (1.5.2) with each  $d_i$  a total divisor of  $d_{i-1}$  ( $i > 1$ ). This gives a pair of decompositions

$$(2.1.1) \quad P = \bigoplus_{i=1}^n P_i \quad \text{and} \quad U = \bigoplus_{i=1}^n U_i$$

with each  $P_i \cong \Lambda$  and each  $f(P_i) = U_i$ . Moreover, since each  $d_i$  is a total divisor of  $d_{i-1}$  the module  $U' = \bigoplus_{i=2}^n U_i$  is annihilated by  $d_1$ , hence is unfaithful.

Now suppose that  $P$  has rank  $\geq 3$ . Then, by hypothesis,  $U'$  is uniquely presentable by  $P' = \bigoplus_{i=2}^n P_i$ . Hence applying Corollary 1.2 to the presentation

$$(2.1.2) \quad f: P = P_1 \oplus P' \rightarrow U = U_1 \oplus U',$$

shows that  $U$  is uniquely presented by  $P$ , as desired.

Unfortunately, this simple argument fails when  $n = 2$  because  $P'$  then has rank 1. The only way out seems to be to modify the proof of the Lifting and Straightening

Theorem, which we do in the next lemma. To apply the lemma, recall that every  $\Lambda$ -module of finite length is the direct sum of a completely faithful module and an unfaithful module [ER '70, 3.9].  $\square$

**2.2 LEMMA.** *Let  $f: P \rightarrow U = V \oplus W$  be a presentation of a  $\Lambda$ -module, with  $P$  free of rank 2,  $U$  of finite length,  $V$  completely faithful, and  $W$  unfaithful. Suppose that  $W$  is uniquely presentable by  $P$ . Then  $U$  is uniquely presentable by  $P$ .*

**PROOF.** Let  $P = \Lambda p_1 \oplus \Lambda p_2$ . Since  $P$  can be mapped onto  $W$ , the diagonalization argument in (1.5.2) gives a decomposition  $W = \Lambda w_1 \oplus \Lambda w_2$ .

Recall from [LR '74, Lemma 1.11] that, over any ring, if  $S$  is a projective module not of finite length and  $S$  can be mapped onto a module  $L$  of finite length, and  $C$  is any completely faithful module of finite length, then  $S$  can be mapped onto  $C \oplus L$ .

Applying this with  $S = \Lambda$  and  $L = 0$ , we see that  $\Lambda$  can be mapped onto  $V$ , say  $V = \Lambda v$ . A second application of this lemma shows that  $\Lambda$  can be mapped onto  $\Lambda v \oplus \Lambda w_1$ . Applying the Lifting and Straightening Theorem to the presentation

$$(2.2.1) \quad f: P = \Lambda p_1 \oplus \Lambda p_2 \rightarrow U = (\Lambda v \oplus \Lambda w_1) \oplus \Lambda w_2$$

we get a new decomposition  $P = \Lambda p_1 \oplus \Lambda p_2$  such that  $f(\Lambda p_1) = \Lambda v \oplus \Lambda w_1$  and  $f(\Lambda p_2) = \Lambda w_2$ . After a change of notation we get

$$(2.2.2) \quad f(p_1) = v + w_1 \quad \text{and} \quad f(p_2) = w_2.$$

Let  $\pi_v$  and  $\pi_W$  be the projection maps from  $P$  to  $\Lambda v$  and  $W$  respectively.

Now consider another presentation  $g: S \rightarrow X$  with  $S \cong P$  and  $X \cong U$ . We show that  $f \sim g$  by obtaining decompositions  $f \sim f_1 \oplus f_2$  and  $g \sim g_1 \oplus g_2$  in which each  $f_i \sim g_i$ .

Since  $X \cong U$ , there is a decomposition  $X = \Lambda y \oplus Z$  with  $\Lambda y \cong \Lambda v$  and  $Z \cong W$ . Since we are assuming that  $W$  is uniquely presentable by  $P$ , we have

$$(2.2.3) \quad (\pi_W f: P \rightarrow W) \sim (\pi_Z g: S \rightarrow Z).$$

This yields decompositions  $S = \Lambda s_1 \oplus \Lambda s_2$  and  $Z = \Lambda z_1 \oplus \Lambda z_2$  such that

$$(2.2.4) \quad (\pi_W f: \Lambda p_i \rightarrow \Lambda w_i) \sim (\pi_Z g: \Lambda s_i \rightarrow \Lambda z_i) \quad (i = 1, 2)$$

where each  $s_i \rightarrow z_i$  and

$$(2.2.5) \quad \text{ann}(w_1) = \text{ann}(z_1) = (\text{say})K.$$

So we have  $\ker(\pi_W f \mid \Lambda p_1) = K p_1$  and  $\ker(\pi_Z g \mid \Lambda s_1) = K s_1$ .

Now we make a sequence of basis changes in  $S$ . The basis changes will be *elementary*, replacing  $s_1$  by an expression of the form  $s_1 + t s_2$  or replacing  $s_2$  by  $s_2 + t s_1$  where  $t$  is an element of the annihilator  $T$  of  $Z$ . Taking  $t \in T$  guarantees that (2.2.4) still holds after the old  $s_i$  is replaced by the new  $s_i$ , because  $\pi_Z g(s_i)$  remains unchanged.

We have  $T \neq 0$  since  $Z$  is unfaithful. The following fact will be used repeatedly.

$$(2.2.6) \quad T y' = \Lambda y' \quad (\forall y' \in \Lambda y).$$

Clearly  $T y' \subseteq \Lambda y'$ . If equality did not hold, then the nonzero module  $\Lambda y' / T y'$  would be annihilated by the nonzero ideal  $T$ , contrary to complete faithfulness of  $\Lambda y$ .

Our first claim is that there is a basis change  $s_2 \rightarrow s_2 + ts_1$  that achieves  $\pi_y g(\Lambda s_2) = \Lambda y$ . Applying (3) in the proof of the Building Lemma [LR, 1.3] to the surjection  $\pi_y g: \Lambda s_2 \oplus \Lambda s_1 \rightarrow \Lambda y$  gives a homomorphism  $\vartheta: \Lambda s_2 \rightarrow \Lambda s_1$  such that  $\pi_y g(1 + \vartheta)(\Lambda s_2) = \Lambda y$ . We have  $\vartheta(s_2) = ts_1$  for some  $t \in \Lambda$ . We then have  $\Lambda y = \Lambda \pi_y g(s_2 + ts_1)$  as needed. Moreover,  $t\pi_y g(s_1) \in \Lambda y$  so, by (2.2.6), we can take  $t \in T$  as claimed.

Since  $\pi_y g(\Lambda s_2) = \Lambda y$  we can replace  $y$  by a different element of  $\Lambda y$  to achieve  $\pi_y g(s_2) = y$ .

Next we claim that there is a basis change  $s_1 \rightarrow s_1 + ts_2$  that achieves  $\pi_y g(s_1) = 0$ . For some  $t \in \Lambda$  we have  $\pi_y g(s_1) = -ty = -t\pi_y g(s_2)$  as needed. Again, by (2.2.6), we can take  $t \in T$ .

Now that we have “erased” the unknown image of  $\Lambda s_1$  in  $\Lambda y$ , we rebuild this image to suit our needs.

Since  $\Lambda v \cong \Lambda y$ , there is an element  $t \in \Lambda$  such that  $\Lambda ty = \Lambda y$  and  $\text{ann}(ty) = \text{ann}(v)$ . Again we can take  $t \in T$ . After replacing  $s_1$  by  $s_1 + ts_2$  we have  $\pi_y g(\Lambda s_1) = \Lambda ty = \Lambda y$ .

After replacing  $y$  by  $ty$  we have  $\pi_y g(s_1) = y$  and  $\text{ann}(y) = \text{ann}(v) = (\text{say}) H$ . Therefore, by (2.2.5),

$$\text{ann}(y + z_1) = \text{ann}(y) \cap \text{ann}(z_1) = H \cap K = \text{ann}(v + w_1).$$

Since  $g(s_1) = y + z_1$  we now have  $\ker(g | \Lambda s_1) = (H \cap K)s_1$  and  $\ker(f | \Lambda p_1) = (H \cap K)p_1$ . Therefore

$$(2.2.7) \quad (g | \Lambda s_1) \sim (f | \Lambda p_1).$$

We no longer need the image of  $\Lambda s_2$  in  $\Lambda y$ . So, by means of one more basis change of the form  $s_2 \rightarrow s_2 + ts_1$  we erase it, replacing it with 0.

Setting  $i = 2$  in (2.2.4), and observing that the projection maps  $\pi$  now have no effect, we get  $(g | \Lambda s_2) \sim (f | \Lambda p_2)$  which, together with (2.2.7), completes the proof of the lemma.  $\square$

### 3. Unfaithful case.

3.1 *Notation.* Let  $f, g: P = \Lambda^n \rightarrow U$  be presentations of an unfaithful left  $\Lambda$ -module of finite length, with  $\ker(f) = K$  and  $\ker(g) = L$ . We have  $TU = 0$  for some nonzero 2-sided ideal  $T$  of  $\Lambda$ . Let  $E(P)$  denote the endomorphism ring of  $P$ . We define

$$(3.1.1) \quad \text{hom}(f, g) = \{\varphi \in E(P) \mid K\varphi \subseteq L\} = \{\varphi \in E(P) \mid (\exists \vartheta \in E(U))\varphi g = f\vartheta\}.$$

In particular, we let  $\text{hom}(f, f) = E(f) = E$ . Note that  $\text{hom}(f, g)$  is a left  $E$ -module.

Let  $\Gamma = \Lambda_{n \times n}$ . We write elements of  $P$  as rows. So every element of  $E(P)$  becomes right multiplication by a unique element of  $\Gamma$ , and we make the identification  $E(P) = \Gamma$ . Thus  $E = E(f)$  is a subring of  $\Gamma$ . Let  $\mathcal{F} = T_{n \times n}$ . Since  $TU = 0$ , we have

$$(3.1.2) \quad \mathcal{F} \subseteq H \quad \text{and} \quad \mathcal{F} \subseteq E.$$

In fact,  $\mathcal{F}$  is a 2-sided ideal of both  $E$  and  $\Gamma$ .

For any  $\lambda$ -module homomorphism  $h: S \rightarrow V$  we define the homomorphism  $f \oplus h: P \oplus S \rightarrow U \oplus V$  by  $(p, s) \rightarrow (pf, sh)$ . It therefore makes sense to speak of the category  $\text{div}(f)$  of all direct summands of the maps  $f^n$  ( $n = 1, 2, \dots$ ).

**3.2 DRESS'S LEMMA.** *The functor  $\text{hom}(f, \dots)$  is a category equivalence between the  $\text{div}(f)$  and  $\text{div}(E) = \{\text{finitely generated projective left } E\text{-modules}\}$ . In particular,  $g \sim f$  if and only if  $\text{hom}(f, g) \cong E$  as left  $E$ -modules.*

**PROOF.** Dress's well-known observation [D '69, p. 985] is that, if  $F$  is a left module over a ring  $R$ , then the functor  $\text{hom}(F, \dots)$  is a category equivalence between  $\text{div}(F)$  and  $\text{div}(E(F))$ , the inverse functor being  $F \otimes_{E(F)} (\dots)$ .

Let  $T_2(\Lambda)$  be the ring of  $2 \times 2$  upper triangular matrices over  $\Lambda$ . In [GL '88, 1.8] a functor  $f \rightarrow M(f)$  is described that is a category equivalence between the category of homomorphisms of left  $\Lambda$ -modules and a subcategory of the category of left  $T_2(\Lambda)$ -modules. (Actually, in that discussion, homomorphisms act on the left, and lower triangular matrices are used.)

To obtain our lemma from Dress's original version, let  $F = M(f)$  and compose the functor  $M(\dots)$  in the previous paragraph with Dress's functor.  $\square$

**3.3 LEMMA.**  *$E/\mathcal{I}$  is a left and right artinian ring.*

**PROOF.** Note that (by considering Morita equivalence) a ring  $R$  is artinian if and only if, for some  $s$ , the matrix ring  $R_{s \times s}$  is artinian.

So it suffices to prove the lemma with  $E(f^s)$  in place of  $E(f)$ . Choose  $s$  such that the Stable Invariant Functor Theorem 1.4 holds for  $K \subseteq P$ . After the change of notation that replaces  $f$  by  $f^s$ , the decomposition given by the Stable Invariant Factor Theorem takes the form  $P = \Lambda^n$  and  $K = \bigoplus_{i=1}^n T_i$  where each  $T_i$  is a 2-sided ideal of  $\Lambda$  containing  $T$ .

The subring  $E$  of  $\Lambda_{n \times n}$  consists of all matrices  $\varphi$  such that, for all  $(i, j)$ , we have  $T_i \varphi_{ij} \subseteq T_j$ . Since each  $T_i$  is a 2-sided ideal, each subset  $E_{ij}$  of  $E$  is therefore a 2-sided ideal of  $\Lambda$  containing  $T$ , and  $E = \sum_{ij} E_{ij}$ .

It follows that  $E$  is a finitely generated  $\Lambda$ -module on the left and on the right. Hence  $E/\mathcal{I}$  is a finitely generated left and right module over the artinian ring  $\Lambda/T$ . In particular,  $E/\mathcal{I}$  is an artinian ring.  $\square$

**3.4 THEOREM.** *Any two presentations  $f, g: P = \Lambda^n \rightarrow U$ , where  $U$  is unfaithful of finite length, and  $n \geq 2$ , are equivalent.*

**PROOF.** By Theorem 1.5,  $U^s$  is uniquely presentable by  $P^s$  for some  $s$ . Therefore  $f^s \sim g \oplus f^{s-1}$ . In particular,  $g \in \text{div}(f)$ . Applying the functor in Dress's Lemma, we get an isomorphism of projective left  $E$ -modules:

$$(3.4.1) \quad E^s \cong H \oplus E^{s-1} \quad \text{where } H = \text{hom}(f, g).$$

To complete the proof of the theorem, it now suffices to show  $E \cong H$ .

By (3.1.2) we can set  $\bar{H} = H/\mathcal{I}$  and  $\bar{E} = E/\mathcal{I}$ . We also set  $\bar{\Gamma} = \Gamma/\mathcal{I}$ .

Let  $\bar{\Lambda} = \Lambda/T$ . Since free modules are projective, there is a natural identification  $\bar{H} = \text{hom}(\bar{f}, \bar{g})$  where  $\bar{f}, \bar{g}: \bar{P} = \bar{\Lambda}^n \rightarrow U$  are the presentations of the  $\bar{\Lambda}$ -module  $U$  induced by  $f$  and  $g$ . Applying this to the situation  $f = g$ , we see that we can also make the identification  $\bar{E} = E(\bar{f})$ . Since the ring  $\bar{\Lambda}$  is a homomorphic image of a PID,  $\bar{\Lambda}$  is artinian, and therefore has 1 in its stable range. Over a ring with 1 in its stable range, every module is uniquely presentable by every free module that presents it, by [W '78, Theorem 4; or G '82, 3.1]. So  $\bar{f} \sim \bar{g}$ . Hence  $\bar{E} \cong \bar{H}$  as left  $\bar{E}$ -modules. In particular,  $\bar{H} = \bar{E}\bar{\alpha}$  for some  $\bar{\alpha} \in \bar{H}$ . Since  $\bar{f} \sim \bar{g}$ , any such  $\bar{\alpha}$  is an

element of  $E(\overline{P})^* = \overline{\Gamma}^*$ , where  $*$  denotes “units of”. We claim

$$(3.4.2) \quad \Gamma = \Gamma H = \Gamma E.$$

Since  $\overline{H} = \overline{E}\overline{\alpha}$  we have  $H = \mathcal{S} + E\alpha$ . So  $\overline{\Gamma}\overline{H} = \overline{\Gamma}\overline{\alpha}$  = (since  $\overline{\alpha} \in \overline{\Gamma}^*$ )  $\overline{\Gamma}$ . Since both  $\Gamma$  and  $\Gamma H$  contain  $\mathcal{S}$ , the first equality in (3.4.2) follows. The second equality follows from the first one by setting  $f = g$ .

Conversely, let  $\overline{\alpha} \in \overline{H} \cap \overline{\Gamma}^*$ . Then  $\overline{H} = \overline{E}\overline{\alpha}$ . We claim

$$(3.4.3) \quad H \cong E \Leftrightarrow \overline{\alpha} \in \overline{E}^* \Gamma^* \quad (\subseteq \overline{\Gamma}^*).$$

The product  $\overline{E}^* \Gamma^*$  makes sense if we reduce the factor in  $\Gamma$  modulo the 2-sided ideal  $\mathcal{S}$  before multiplying.

Suppose  $H \cong E$ . Then  $H = E\beta$  for some  $\beta \in E$ . We have  $\beta \in \Gamma^*$  by (3.4.2). Then  $\overline{H} = \overline{E}\beta = \overline{E}\overline{\alpha}$  shows  $\overline{\alpha}\beta^{-1} \in \overline{E}^*$  as desired. Conversely, suppose  $\overline{\alpha} = \overline{\varphi}\beta$  with  $\overline{\varphi} \in \overline{E}^*$  and  $\beta \in \Gamma^*$ . Then  $\overline{H} = \overline{E}\overline{\alpha} = \overline{E}\beta$  so  $H = E\beta$ . (Note that both  $H$  and  $E$  contain  $\mathcal{S}$ , and since  $\beta \in \Gamma^*$ , so does  $E\beta$ .) Hence  $H = E\beta \cong E$ .

Now we prove that  $H \cong E$  by verifying the right-hand side of (3.4.3).

Let  $\alpha \in \overline{H} \cap \overline{\Gamma}^*$  so  $\overline{H} = \overline{E}\overline{\alpha}$ . Then  $A = (\overline{\alpha}, 1, 1, \dots, 1)$  is an isomorphism:  $\overline{E}^s \cong \overline{H} \oplus \overline{E}^{s-1}$ . In view of (3.4.1) we can apply the version of (3.4.3) that applies to the presentation  $f^s: P^s \rightarrow U^s$ , getting

$$(3.4.4) \quad A = (\overline{\alpha}, 1, 1, \dots, 1) \in GL_s(\overline{E})GL_s(\Gamma), \quad \text{say } A = xy.$$

In what follows, we repeatedly use the following two facts about  $\mathbf{K}_1$ . If a ring  $R$  has  $m$  in its stable range, then the natural map:  $GL_m(R) \rightarrow \mathbf{K}_1(R)$  is a surjection, and the kernel of the map:  $GL_{m+1}(R) \rightarrow \mathbf{K}_1(R)$  is the subgroup generated by elementary matrices.

Let  $\nu_\Lambda$  denote “natural image in  $\mathbf{K}_1(\Lambda)$ ”. The PID  $\Lambda$  has Krull dimension 1, therefore, by [S ’77, 2.1] has 2 in its stable range. So  $\nu_\Lambda(y) = \nu_\Lambda(\gamma)$  for some  $\gamma \in GL_2(\Lambda)$ . Since  $P$  has rank  $\geq 2$ , we have  $GL_2(\Lambda) \subseteq \Gamma^*$ , so  $\gamma \in \Gamma^*$ . Let  $\nu'$  denote “natural image in  $\mathbf{K}_1(\overline{E})$ ”. Since the artinian ring  $\overline{E}$  has 1 in its stable range, we have  $\nu'(x) = \nu'(\overline{\varphi})$  for some  $\overline{\varphi} \in \overline{E}^*$ . Letting  $\nu$  denote “natural image in  $\mathbf{K}_1(\overline{\Gamma})$ ”, we get  $\nu(\overline{\alpha}) = \nu(\overline{\varphi})\nu(\gamma) = \nu(\overline{\varphi}\gamma)$  where we compute  $\overline{\varphi}\gamma$  by viewing  $\gamma$  as a matrix over  $\Lambda$ , and reducing its entries modulo  $T$ .

Since 1 is in the stable range of  $\overline{\Gamma}$ , and  $\overline{\Gamma}$  consists of matrices of size at least  $2 \times 2$  over  $\overline{\Lambda}$ , we now have  $\overline{\alpha} = \overline{\varphi}\gamma\overline{\varepsilon}$  where  $\overline{\varepsilon}$  is a product of elementary matrices over  $\overline{\Lambda}$ , hence can be lifted to a unit  $\varepsilon \in \Gamma^*$ . The relation  $\overline{\alpha} = \overline{\varphi} \cdot (\gamma\varepsilon)$  now shows, by (3.4.3), that  $H \cong E$ .  $\square$

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