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GROTHENDIECK GROUPS OF ALGEBRAS WITH NILPOTENT ANNIHILATORS

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ABSTRACT. Let R be a commutative noetherian ring and $i\colon R \to \Lambda$ an R-algebra such that Λ is a finitely generated R-module. Then the annihilator of Λ in R is nilpotent if and only if the cokernel of the induced map of Grothendieck groups $i^*\colon K_0(\text{mod } \Lambda) \to K_0(\text{mod } R)$ is a torsion group.

Let k be an algebraically closed field, and let $S = k[[X_1, \ldots, X_n]]$ be the formal power series ring in n variables over k. Let G be a finite subgroup of $\text{GL}(n,k)$. Hence G acts as a group of k-automorphisms of S, and we denote by R the fixed ring $R = S^G$. Denote by $K_0(\text{mod } R)$ the Grothendieck group of the category of finitely generated R-modules mod R modulo exact sequences. In [**1**] we proved that $K_0(\text{mod } R)$ is finitely generated. Under the additional assumption that G acts freely we proved that $K_0(\text{mod } R)$ is isomorphic to $Z[R] \amalg H$, where H is a finite group and $[R]$ denotes the image of R in $K_0(\text{mod } R)$. The motivation for this paper was to show that the assumption that G acts freely is not necessary. The proof we give here is based on the proof given in the Bielefeld May 1985 conference on representation theory and singularity theory. A different proof in the case that G is abelian has been given in [**3**]. Our desired result is an easy consequence of the following general result which is also of independent interest. For the rest of the paper R denotes a commutative noetherian ring and Λ an R-algebra via a fixed map $i\colon R \to \Lambda$ such that Λ is a finitely generated R-module.

THEOREM. *The annihilator of* Λ, $\text{ann}_R \Lambda$, *is a nilpotent ideal in R if and only if* $\text{Coker}(K_0(\text{mod } \Lambda) \xrightarrow{i^*} K_0(\text{mod } R))$ *is a torsion group.*

This result is a direct consequence of the following two propositions.

PROPOSITION 1. *If* $\text{ann}_R \Lambda$ *is nilpotent, then*

$$\text{Coker}(K_0(\text{mod } \Lambda) \xrightarrow{i^*} K_0(\text{mod } R))$$

is torsion.

PROOF. Now, $\text{ann}_R \Lambda$ is nilpotent if and only if $\Lambda_p \neq 0$ for all prime ideals p in R. From this characterization it follows easily that if $\mathfrak{A}$ is an ideal in R and we consider the $R/\mathfrak{A}$-algebra $R/\mathfrak{A} \to \Lambda/\mathfrak{A}\Lambda$, then $\text{ann}_{R/\mathfrak{A}} \Lambda/\mathfrak{A}\Lambda$ is nilpotent. Assume that the proposition is false. Then because R is noetherian there is an ideal $\mathfrak{A}$ in R such that $\text{Coker}(K_0(\text{mod } \Lambda/\mathfrak{A}\Lambda) \to K_0(\text{mod } R/\mathfrak{A}))$ is not torsion, and

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$\operatorname{Coker}(K_0(\operatorname{mod} \Lambda/b\Lambda) \to K_0(\operatorname{mod} R/b))$ is torsion for any ideal $\mathfrak{B}$ in R properly containing $\mathfrak{A}$. We can clearly assume that $\mathfrak{A}$ is zero.

Consider for a minimal prime ideal p in R the commutative exact diagram [**2**, p. 642]

$$\begin{array}{ccccccc}
\lim_{t\notin p} K_0(\operatorname{mod} \Lambda/t\Lambda) & \longrightarrow & K_0(\operatorname{mod} \Lambda) & \longrightarrow & K_0(\operatorname{mod} \Lambda_p) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \beta} & & \\
\lim_{t\notin p} K_0(\operatorname{mod} R/tR) & \longrightarrow & K_0(\operatorname{mod} R) & \longrightarrow & K_0(\operatorname{mod} R_p) & \longrightarrow & 0
\end{array}$$

The snake lemma gives the exact sequence $\operatorname{Coker}\alpha \to \operatorname{Coker} i^* \to \operatorname{Coker}\beta \to 0$. Since R_p is a local artin ring, $K_0(\operatorname{mod} R_p) \simeq Z$, and since $\Lambda_p \neq 0$, β is not zero, so that $\operatorname{Coker}\beta$ is torsion. By our assumption $\operatorname{Coker}(K_0(\operatorname{mod} \Lambda/t\Lambda) \to K_0(\operatorname{mod} R/tR))$ is torsion since $t \neq 0$, and hence $\operatorname{Coker}\alpha$ is torsion. From this it follows that $\operatorname{Coker} i^*$ is torsion, which contradicts our assumption, and the proof is done.

PROPOSITION 2. *We have* $\operatorname{rank}(\operatorname{Coker} K_0(\operatorname{mod} \Lambda \xrightarrow{i^*} K_0(\operatorname{mod} R)) \geq n$, *where n denotes the number of minimal primes in R not containing* $\mathfrak{A} = \operatorname{ann}_R \Lambda$.

PROOF. Let $p_1, \ldots, p_n, \ldots, p_r$ be the minimal primes in R, where $\mathfrak{A} \subsetneq p_i$ for $1 \leq i \leq n$ and $\mathfrak{A} \subset p_i$ for $n < i \leq r$. Let $T = R \setminus \bigcup_{i=1}^{r} p_i$, and consider the diagram

$$\begin{array}{ccc}
K_0(\operatorname{mod} R/\mathfrak{A}) & \longrightarrow & K_0(\operatorname{mod}(R/\mathfrak{A})_T) \\
\downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \gamma} \\
K_0(\operatorname{mod} R) & \longrightarrow & K_0(\operatorname{mod} R_T)
\end{array}$$

Since R_T is artin, $K_0(\operatorname{mod} R_T)$ is a free group with basis $\{[(R/p_i)_T];\, 1 \leq i \leq r\}$ and since $(R/\mathfrak{A})_T$ is artin $K_0(\operatorname{mod}(R/\mathfrak{A})_T)$ is a free group with basis $\{[(R/p_i)_T]; r \leq i \leq r\}$. Hence we have $\operatorname{rank} \operatorname{Coker}\delta \geq \operatorname{rank} \operatorname{Coker}\gamma \geq n$. Since $i\colon R \to \Lambda$ has the factorization $R \to R/\mathfrak{A} \to \Lambda$, we get $\operatorname{rank} \operatorname{Coker} i^* \geq \operatorname{rank} \operatorname{Coker}\delta \geq n$.

We end this note with the following consequences of Proposition 1.

COROLLARY 3. *If Λ is a commutative semilocal regular domain and $R \subset \Lambda$, then* $\operatorname{rank} K_0(\operatorname{mod} R) = 1$.

PROOF. Since Λ is regular, $K_0(\operatorname{mod} \Lambda) \simeq K_0(\mathscr{P}(\Lambda), 0)$, where $\mathscr{P}$ denotes the category of finite generated projective Λ-modules and $K_0(\mathscr{P}, 0)$ denotes the free group on the isomorphism classes of objects in $\mathscr{P}$ modulo split exact sequences. Then $K_0(\operatorname{mod} \Lambda) \simeq Z$ since every finitely generated projective Λ-module is free. Since $R \subset \Lambda$, R is also a domain. If L denotes the quotient field of R, we have a surjection $K_0(\operatorname{mod} R) \to K_0(\operatorname{mod} L)$, so that $\operatorname{rank} K_0(\operatorname{mod} R) \geq 1$. It then follows from Proposition 1 that $\operatorname{rank} K_0(\operatorname{mod} R) = 1$.

COROLLARY 4. *Let Λ be a commutative complete regular local domain and G a finite group acting on Λ as ring automorphisms, such that the order of G is invertible in Λ. Then $K_0(\operatorname{mod} \Lambda^G) \simeq Z \oplus H$, where H is a finite group.*

PROOF. Λ is a finitely generated Λ^G-module [**4**, Corollary 5.9] and $R = \Lambda^G$ is noetherian [**4**, Corollary 1.12]. Then $\operatorname{rank} K_0(\operatorname{mod} \Lambda^G) = 1$ follows from Corollary 3, and it only remains to show that $K_0(\operatorname{mod} \Lambda^G)$ is finitely generated. This follows as in [**1**, Proposition 3.4]: Since $\mathrm{End}_{\Lambda G}(\Lambda) \simeq \Lambda^G$, there is a surjection $K_0(\mathrm{mod}\ \Lambda G) \to K_0(\mathrm{mod}\ \Lambda^G)$. Further ΛG has finite global dimension since the order of G is invertible in Λ, and hence $K_0(\mathrm{mod}\ \Lambda G) \simeq K_0(\mathscr{P}(\Lambda G), 0)$. Since by our assumption on Λ the Krull-Schmidt property holds for ΛG, $K_0(\mathscr{P}(\Lambda G), 0)$ is finitely generated.

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