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THE NUMBER OF GENERATORS OF MODULES OVER POLYNOMIAL RINGS

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ABSTRACT. Let k be an infinite field and $B = k[X_1, \ldots, X_n]$ a polynomial ring over k. Let M be a finitely generated module over B. For every prime ideal $P \subset B$ let $\mu(M_P)$ be the minimum number of generators of M_P, i.e., $\mu(M_P) = \dim_{B_P/P_P}(M_P \otimes_{B_P} (B_P/P_P))$. Set $\eta(M) = \max\{\mu(M_P) + \dim(B/P) | P \in \operatorname{Spec} B$ such that M_P is not free$\}$. Then M can be generated by $\eta(M)$ elements. This improves earlier results of A. Sathaye and N. Mohan Kumar on a conjecture of Eisenbud-Evans.

Given a finitely generated module M over a commutative ring B, one would like to determine $\mu(M)$, the minimum number of generators of M.

This problem goes back to Serre (as mentioned in [**Sw2**]).

Of course, if B is local, $\mu(M) = \dim_k(M \otimes_B k)$, where k is the residue field of B.

If B is not local, for every prime ideal $P \subset B$ one sets

$$b_P(M) = \begin{cases} \mu(M_P) + \dim(B/P) & \text{if } M_P \neq 0, \\ 0 & \text{if } M_P = 0. \end{cases}$$

The Forster-Swan theorem [**F**, **Sw1**], states that $\mu(M)$ is less than or equal to the maximum of $b_P(M)$ as P runs through all j-primes of B. A j-prime is a prime which is an intersection of maximal ideals.

If $B = A[X]$, A. Sathaye [**Sa**] and N. Mohan Kumar [**MK**] proved that M can be generated by $\max\{b_P(M) | P \in \operatorname{Spec} B$ such that $\dim(B/P) < \dim B\}$ elements.

In this short note we considerably improve the Sathaye-Mohan Kumar estimate in the case when B is a polynomial ring in n variables over an infinite field k. Namely we show that in this case it is enough to take the maximum of $b_P(M)$ only over those prime ideals of B at which M is not free (Theorem 2).

For example, if M is not free at a unique maximal ideal at which it can be generated by d elements, our results show that M can be generated by d elements, while the Sathaye-Mohan Kumar results only show that M can be generated by $\max\{d, r + n - 1\}$ elements, where r is the rank of M.

N. Mohan Kumar also proved in the same paper that if $B = k[X_1, \ldots, X_n]$ and $I \subset B$ an ideal such that I/I^2 is d-generated and $d \geq 2 + \dim(B/I)$, then I is d-generated.

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This result of Mohan Kumar is a special case (when $M = I$) of the following

THEOREM 1. *Let $B = k[X_1, \ldots, X_n]$, where k is an infinite field and M a finitely generated B-module of rank r. Let I be the defining ideal of the set of primes of B at which M is not free. Assume M/IM can be generated by d elements. If $d \geq 1 + r + \dim(B/I)$, then M can be generated by d elements.*

PROOF OF THEOREM 1. Let $f_1, \ldots, f_s$ be some generators of I. For every finitely generated B-module N set

$$b(N_{f_i}) = \max\{b_P(N_{f_i}) | P \text{ a prime in } B_{f_i}\}$$

and set

$$b^I(N) = \max\{b(N_{f_i}) | i = 1, \ldots, s\}.$$

Clearly

$$b^I(N) = \max\{b_P(N) | P \in \operatorname{Spec} B, P \not\supset I\}.$$

Thus, $b^I(M) = r + n$.

Let $a_1, \ldots, a_d$ be elements of M which generate M/IM. We claim there exist $a'_1, \ldots, a'_d \in M$ such that they also generate M/IM and $b^I(M/(Ba'_1 + \cdots + Ba'_d)) \leq b^I(M) - d = r + n - d$.

For every t set $N_t = M/(Ba'_1 + \cdots + Ba'_t)$.

The elements $a'_1, \ldots, a'_d$ will be constructed by induction on t subject to the condition that N_t/IN_t be generated by $(d - t)$ elements and $b^I(N_t) \leq b^I(M) - t$ for every t.

Assume $a'_1, \ldots, a'_t$ have been found and let $a_{t+1} \in M$ form part of a generating set of N_t/IN_t consisting of $d - t$ elements. Then any element $a'_{t+1} = a_{t+1} + \lambda y$, where $\lambda \in I$, and $y \in M$ also will form part of a system of generators of N_t/IN_t consisting of $d - t$ elements. Since $N_{t+1}/IN_{t+1} = N_t/(Ba'_{t+1} + IN_t) = (N_t/IN_t)/Ba'_{t+1}$, the module N_{t+1}/IN_{t+1} will be generated by $d - t - 1$ elements.

To ensure that $b^I(N_{t+1}) \leq b^I(N_t) - 1$ it is enough to ensure that $b((N_{t+1})_{f_i}) \leq b((N_t)_{f_i}) - 1$ for every i. It follows from the proof of the main theorem of [**F**] that in order that $b((N_{t+1})_{f_i}) \leq b((N_t)_{f_i}) - 1$ it is necessary and sufficient that the element a'_{t+1} be basic for $(N_t)_{f_i}$ at some finitely many primes of B_{f_i} (namely, at those primes where $b_P((N_t)_{f_i}) = b((N_t)_{f_i})$; there are only finitely many of those). Summarizing, we see that in order that $b^I(N_{t+1}) \leq b^I(N_t) - 1$ it is sufficient to make a'_{t+1} basic for N_t at some finite set of primes $P_1, \ldots, P_v$ of B not one of which contains I.

Let $y \in M$ be an element basic for N_t at $P_1, \ldots, P_v$. It exists by Hilfssatz [**F**]. Since $P_i \not\supset I$ for every i, we can find $\lambda \in I \backslash (P_1 \cup \cdots \cup P_v)$ such that $a'_{t+1} = a_{t+1} + \lambda y$ is basic at $P_1, \ldots, P_v$. This proves the claim.

Set $N = N_d = M/(Ba'_1 + \cdots + Ba'_d)$ and set $J = \operatorname{ann}(N)$. We claim that $\dim(B/I) + \dim(B/J) \leq n - 2$. Indeed, let P be a minimal prime over-ideal of J. Then $\mu(N_P) \geq 1$ and $b_P(N) = \mu(N_P) + \dim(B/P) \leq r + n - d$, which implies $\dim(B/P) \leq r + n - d - \mu(N_P) \leq r + n - d - 1$.

Since $d \geq 1 + r + \dim(B/I)$, we see that $\dim(B/I) \leq d - r - 1$ and therefore $\dim(B/I) + \dim(B/J) \leq r + n - d - 1 + d - r - 1 \leq n - 2$.

Since I belongs to the radical of B_{1+I} and $a'_1, \ldots, a'_d$ generate M/IM, by Nakayama's lemma they generate M_{1+I}. So $N_{1+I} = 0$ and therefore J contains an element from $1 + I$. This implies $I + J = B$.

Let L be the kernel of the map $\phi\colon B^d \to M$ which sends the ith free generator of B^d to a'_i. Let $f \in B$ be an element such that L_f is free over B_f. We claim there exist new variables $x_1, \ldots, x_n$ such that after putting $A = k[x_1, \ldots, x_{n-1}]$ we will have

(1) $(I \cap A) + (J \cap A) = 1$;

(2) f is monic in x_n.

In fact, a generic linear change will do the trick. Indeed, it is well known that any f becomes monic in x_n after a generic linear change.

Now we have to show that a generic projection has the property that $(I \cap A) + (J \cap A) = 1$. But the algebraic set consisting of all lines joining every point of the algebraic set defined by I to every point of the algebraic set defined by J has dimension at most $\dim(B/I) + \dim(B/J) + 1 \leq n-2+1 = n-1$, so, the intersection of this set of lines with the hyperplane at infinity has dimension $\leq n-2$. Since the hyperplane at infinity has dimension $n-1$, it contains points which do not belong to this set and they form a Zarisky open subset. A projection from one of those points will do. Since a generic linear change satisfies (1) and (2), there exists a linear change which satisfies both (1) and (2).

So, let $s \in I \cap A$ and $s' \in J \cap A$ be such that $s + s' = 1$. Then $a'_1, \ldots, a'_d$ generate $M_{s'}$ and the map $\phi_{s'}\colon B^d_{s'} \to M_{s'}$ is surjective. The kernel of $\phi_{s'}$ is $L_{s'}$. Since $s \in I$, we see that $M_{ss'}$ is projective, so $\phi_{ss'}\colon B^d_{ss'} \to M_{s's}$ splits and $L_{ss'}$ is projective too. Since $L_{ss'}$ becomes free after inverting a monic f, it is free of rank $d-r$.

We construct a projective B-module P of rank d and a surjection $P \twoheadrightarrow M$ as follows. Over $B_{s'}$ take $P_{s'}$ to be the free module of rank d and surject it into $M_{s'}$ by $\phi_{s'}\colon B^d_{s'} \twoheadrightarrow M_{s'}$. Over B_s take P_s to be $M_s \oplus B_s^{d-r}$ and surject it onto M_s by a map $\psi\colon M_s \oplus B_s^{d-r} \twoheadrightarrow M_s$ which is the projection onto the first factor. The surjections $\psi_{s'}$ and $\phi_{s's}$ can be patched up over $B_{ss'}$, since they split and their images are isomorphic to $M_{ss'}$ and their kernels are free of rank $d-r$. This patching produces a projective module of rank d which surjects onto M. Since all projectives over B are free, M is d-generated. Q.E.D.

THEOREM 2. *Set $\eta(M) = \max\{b_P(M) | P \in \operatorname{Spec} B$ such that M_P is not free$\}$. Then M can be generated by $\eta(M)$ elements.*

PROOF. Let I be the defining ideal of the set of primes at which M is not free. Let P be a minimal prime over-ideal of I such that $\dim(B/P) = \dim(B/I)$. Since M_P is not free, $\mu(M_P) \geq 1 + r$ and therefore $\eta(M) \geq \mu(M_P) + \dim(B/P) \geq 1 + r + \dim(B/I)$. Since M/IM is a module over B/I, it can be generated by $\max\{\mu((M/IM)_P) + \dim((B/I)/P) | P \in \operatorname{Spec} B/I\}$. But this number equals $\eta(M)$. Now we are done by Theorem 1 with $d = \eta(M)$.

REMARKS. A theorem of D. Quillen and A. Suslin says that if P is a projective module over $B = A[T]$ which becomes free after inverting a monic polynomial f, then P is free.

Our results suggest the following generalization of the Quillen-Suslin theorem to the nonprojective case.

CONJECTURE. Let $B = A[T]$ where A is commutative Noetherian and M a finitely generated B-module such that M_f is B_f-free of rank r where f is monic in T. Let I be the defining ideal of the set of primes of B at which M is not free. If

M/IM can be generated by d elements and $d \geq 1 + r + \dim(B/I)$, then M also can be generated by d elements.

In the special case when M is an ideal of B this has been proven by S. Mandal [**Ma**].

An interesting problem is to determine, for a given finitely generated module M over some commutative ring B, the minimum rank of a B-projective which surjects onto M.

In particular, one can ask the following

QUESTION. When is M the surjective image of a projective of rank $\eta(M)$?

Boratynski [**Bo**] gives an example of a maximal ideal M in a three-dimensional regular affine algebra over the reals which is not the surjective image of a rank 3 projective (although clearly $\eta(M) = 3$). So the answer is not always positive.

On the other hand, Murthy [**Mu**] has shown that the answer is positive for ideals in affine algebras over algebraically closed fields.

PROPOSITION. *Let M be a finitely generated B-module and $s, s' \in B$ elements such that M_s is B_s-projective of rank r and $M_{s'}$ is $(r+1)$-generated. Then M is the surjective image of a projective B-module of rank $r+1$.*

PROOF. Over $B_{s'}$ take a free module of rank $(r+1)$ which surjects onto $M_{s'}$ and over B_s take $M_s \oplus (\bigwedge^r M_s)^*$ with the projection onto the first summand. They patch over $B_{ss'}$ and give a projective of rank $r+1$ over B which surjects onto M.

With the help of this proposition one can show that the answer to the above question is positive provided $\dim B \leq 2$.

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