

**ELEMENTARY EQUIVALENCE AND PROFINITE COMPLETIONS:
A CHARACTERIZATION OF FINITELY GENERATED
ABELIAN-BY-FINITE GROUPS**

FRANCIS OGER

(Communicated by Thomas J. Jech)

ABSTRACT. In this paper, we show that any finitely generated abelian-by-finite group is an elementary submodel of its profinite completion. It follows that two finitely generated abelian-by-finite groups are elementarily equivalent if and only if they have the same finite images. We give an example of two finitely generated abelian-by-finite groups G, H which satisfy these properties while $G \times \mathbf{Z}$ and $H \times \mathbf{Z}$ are not isomorphic. We also prove that a finitely generated nilpotent-by-finite group is elementarily equivalent to its profinite completion if and only if it is abelian-by-finite.

The definitions and results of model theory which are used here, and in particular the notions of elementary equivalence and elementary submodel, are given in [2]. Anyhow, for the sake of brevity, we denote by G^U the ultrapower G^I/U , for each group G and each ultrafilter U over a set I . Concerning group theory, the reader is referred to [9 and 10].

We obtain the following characterization:

THEOREM. *A finitely generated nilpotent-by-finite group is elementarily equivalent to its profinite completion if and only if it is abelian-by-finite.*

This result is a consequence of Theorems 1 and 2 below.

THEOREM 1. *Any finitely generated abelian-by-finite group is an elementary submodel of its profinite completion.*

THEOREM 2. *If G is a finitely generated nilpotent-by-finite, but not abelian-by-finite, group, then there exists an existential sentence, built up from the multiplication symbol, which is false in G and true in the profinite completion of G .*

Theorem 1 generalizes [7, Theorem 1.4], which was only valid for finitely generated finite-by-abelian groups. A finitely generated finite-by-abelian group is abelian-by-finite, since its center is a normal subgroup of finite index according to the result of P. Hall which is mentioned in [10, p. 12]. On the other hand, the finitely presented abelian-by-finite group $\langle x, y; y^2 = 1, y^{-1}xy = x^{-1} \rangle$ is not finite-by-abelian as its center is trivial.

By [11, Theorem 2.1 and 6, Theorem 5.5], the following properties are equivalent for two finitely generated finite-by-abelian groups G, H :

- (1) G and H are elementarily equivalent;

Received by the editors July 16, 1987.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 20A15; Secondary 03C60, 20F18.

- (2) G and H have isomorphic profinite completions;
- (3) G and H have the same finite images;
- (4) $G \times \mathbf{Z} \cong H \times \mathbf{Z}$;
- (5) There exists an integer $n \geq 1$ such that $\times^n G \cong \times^n H$.

The result below generalizes the equivalence of (1), (2) and (3) to finitely generated abelian-by-finite groups. It is a consequence of Theorem 1, [9, Proposition 4, p. 225] and the remark which follows Corollary 3.4 of [6]:

COROLLARY. *The following properties are equivalent for two finitely generated abelian-by-finite groups G, H :*

- (1) G and H are elementarily equivalent;
- (2) G and H have isomorphic profinite completions;
- (3) G and H have the same finite images.

For two finitely generated abelian-by-finite groups G, H , (4) and (5) are equivalent and imply (1), (2) and (3), according to [3, Theorem 1, 4, Theorem 9 and 5, Theorem 1]. Anyhow, contrary to the case of finitely generated finite-by-abelian groups, the converse is false, as we can see from the following proposition.

PROPOSITION. *Let us consider the finitely presented abelian-by-finite groups:*

$$\begin{aligned}
 M &= \langle a, b; a^{25} = 1, b^{-1}ab = a^6 \rangle, \\
 N &= \langle c, d; c^{25} = 1, d^{-1}cd = c^{11} \rangle, \\
 G &= \langle M \times M, y; y^2 = 1, y^{-1}(u, v)y = (v, u) \text{ for any } u, v \in M \rangle, \\
 H &= \langle N \times N, z; z^2 = 1, z^{-1}(u, v)z = (v, u) \text{ for any } u, v \in N \rangle.
 \end{aligned}$$

Then, G and H are elementarily equivalent, but $G \times \mathbf{Z}$ and $H \times \mathbf{Z}$ are not isomorphic.

REMARK. The characterization of the models of the theory of a finitely generated finite-by-abelian group which is given in [7, Theorem 1.3] is not valid for finitely generated abelian-by-finite groups. In fact, the finitely generated abelian-by-finite group $G = \langle x, y; y^2 = 1, y^{-1}xy = x^{-1} \rangle$ has a trivial center and the same property is true for any group which is elementarily equivalent to G .

§§1, 2 and 3 are respectively devoted to the proofs of Theorem 1, Theorem 2 and the proposition.

1. Proof of Theorem 1.

LEMMA 1.1. *If M is a finitely generated nilpotent-by-finite group, then there exists an integer $n \geq 1$ such that M^n is nilpotent and torsion-free.*

PROOF OF LEMMA 1.1. Let us consider a nilpotent normal subgroup N of M such that M/N is finite. There exists an integer $r \geq 1$ such that M^r is contained in N . The nilpotent group N is finitely generated according to [8, Theorem 1.41] and it follows from [10, Exercise 3.10 and Theorem 3.25] that the set $t(N)$ of all torsion elements of N is finite. As M is residually finite by [9, Theorem 1, p. 17], there exists an integer $s \geq 1$ such that $M^s \cap t(N) = \{1\}$. For each integer n which is divisible by r and s , we have $M^n \subset N$ and $M^n \cap t(N) = \{1\}$; so, M^n is nilpotent and torsion-free.

NOTATIONS. For each group M , for each ultrafilter U over a set I , for each $x \in M$ and for each $a \in \mathbf{Z}^U$, we denote by x^a the element of M^U which admits

$(x^{a(i)})_{i \in I}$ as a representative, where $(a(i))_{i \in I}$ is a representative of a in \mathbf{Z}^I . The element x^a does not depend on the choice of the representative $(a(i))_{i \in I}$.

For each residually finite group M , for each $x \in M$ and for each $a \in \widehat{\mathbf{Z}}$, if $(a(n))_{n \in \mathbf{N}}$ is a sequence of elements of \mathbf{Z} which converges to a in $\widehat{\mathbf{Z}}$, then $(x^{a(n)})_{n \in \mathbf{N}}$ converges in \widehat{M} to an element which we denote by x^a . This element does not depend on the choice of the sequence $(a(n))_{n \in \mathbf{N}}$.

PROOF OF THEOREM 1. We must show that, if G is a finitely generated abelian-by-finite group, then G and \widehat{G} satisfy the same sentences with parameters in G . According to [2, Theorem 4.1.9], it is enough to prove that there exist an ultrafilter U and an isomorphism $f: G^U \rightarrow \widehat{G}^U$ which fixes the elements of G .

By Lemma 1.1, there exists an integer $p \geq 1$ such that G^p is abelian and torsion-free. We consider a basis $\{z_1, \dots, z_s\}$ of the free \mathbf{Z} -module G^p and representatives y_1, \dots, y_r of the elements of G/G^p in G . Each element of G can be written in a unique way $x = y_a z_1^{a(1)} \dots z_s^{a(s)}$ with $1 \leq a \leq r$ and $a(1), \dots, a(s) \in \mathbf{Z}$.

The multiplication law of G is determined by a set of equalities:

$$[z_i, z_j] = 1 \quad \text{for } 1 \leq i, j \leq s,$$

$$y_i y_j = y_{g(i,j)} z_1^{g(1,i,j)} \dots z_s^{g(s,i,j)} \quad \text{for } 1 \leq i, j \leq r,$$

$$y_i^{-1} z_j y_i = z_1^{h(1,i,j)} \dots z_s^{h(s,i,j)} \quad \text{for } 1 \leq i \leq r \text{ and } 1 \leq j \leq s,$$

with $1 \leq g(i, j) \leq r$ for $1 \leq i, j \leq r$, $g(1, i, j), \dots, g(s, i, j) \in \mathbf{Z}$ for $1 \leq i, j \leq r$ and $h(1, i, j), \dots, h(s, i, j) \in \mathbf{Z}$ for $1 \leq i \leq r$ and $1 \leq j \leq s$. We have $y_i^{-1} z_j^a y_i = (y_i^{-1} z_j y_i)^a = z_1^{h(1,i,j)a} \dots z_s^{h(s,i,j)a}$ for $1 \leq i \leq r$, $1 \leq j \leq s$ and $a \in \mathbf{Z}$.

A sequence $(x_n)_{n \in \mathbf{N}}$ of elements of G , with $x_n = y_{a(n)} z_1^{a(1,n)} \dots z_s^{a(s,n)}$ for each integer n , converges in \widehat{G} if and only if, first, $(a(n))_{n \in \mathbf{N}}$ is stationary and, second, for each integer $1 \leq i \leq s$, $(a(i, n))_{n \in \mathbf{N}}$ converges in $\widehat{\mathbf{Z}}$. In fact, if $(x_n)_{n \in \mathbf{N}}$ converges in \widehat{G} , then, for each integer $q \geq 1$, there exists an integer m such that, for each integer $n \geq m$, $x_n x_m^{-1}$ belongs to $G^{p^q} \subset (G^p)^q$; for each integer $n \geq m$, we have $a(n) = a(m)$ and $a(1, n) - a(1, m), \dots, a(s, n) - a(s, m)$ belong to $q\mathbf{Z}$.

So the profinite completion \widehat{G} of G is equal to the set of all elements $x = y_a z_1^{a(1)} \dots z_s^{a(s)}$ with $1 \leq a \leq r$ and $a(1), \dots, a(s) \in \widehat{\mathbf{Z}}$, where the multiplication law is given by the following rules:

- (1) $z_i^a z_i^b = z_i^{a+b}$ for $1 \leq i \leq s$ and $a, b \in \widehat{\mathbf{Z}}$;
- (2) $[z_i^a, z_j^b] = 1$ for $1 \leq i, j \leq s$ and $a, b \in \widehat{\mathbf{Z}}$;
- (3) $y_i y_j = y_{g(i,j)} z_1^{g(1,i,j)} \dots z_s^{g(s,i,j)}$ for $1 \leq i, j \leq r$;
- (4) $y_i^{-1} z_j^a y_i = z_1^{h(1,i,j)a} \dots z_s^{h(s,i,j)a}$ for $1 \leq i \leq r$, $1 \leq j \leq s$ and $a \in \widehat{\mathbf{Z}}$.

For each ultrafilter U over a set I and for each $x \in G^U$ (respectively \widehat{G}^U), let us consider a representative $(x_i)_{i \in I}$ of x in G^I (respectively \widehat{G}^I). Each element x_i can be written in a unique way $x_i = y_{a(i)} z_1^{a(1,i)} \dots z_s^{a(s,i)}$ with $1 \leq a(i) \leq r$ and $a(1, i), \dots, a(s, i) \in \mathbf{Z}$ (respectively $\widehat{\mathbf{Z}}$). There exists a unique element $a \in \{1, \dots, r\}$ such that $\{i \in I | a(i) = a\}$ belongs to U . We have $x = y_a z_1^{a(1)} \dots z_s^{a(s)}$, where $a(1), \dots, a(s)$ are the elements of \mathbf{Z}^U (respectively $\widehat{\mathbf{Z}}^U$) which admit $a(1, i)_{i \in I}, \dots, a(s, i)_{i \in I}$ as representatives. This decomposition of x is

necessarily unique and does not depend on the choice of the representative $(x_i)_{i \in I}$ of x in G^I (respectively \widehat{G}^I).

So, for each ultrafilter U over a set I , G^U (respectively \widehat{G}^U) is the set of all elements $x = y_a z_1^{a(1)} \cdots z_s^{a(s)}$ with $1 \leq a \leq r$ and $a(1), \dots, a(s) \in \mathbf{Z}^U$ (respectively $\widehat{\mathbf{Z}}^U$), where the multiplication law is obtained by replacing $\widehat{\mathbf{Z}}$ by \mathbf{Z}^U (respectively $\widehat{\mathbf{Z}}^U$) in the rules (1), (2), (3), (4) above; we prove these rules in G^U (respectively \widehat{G}^U) by applying the same rules to representatives in G^I (respectively \widehat{G}^I) of the elements that we consider.

As \mathbf{Z} is an elementary submodel of its profinite completion, we obtain elementarily equivalent structures by interpreting, in \mathbf{Z} and $\widehat{\mathbf{Z}}$, each element of \mathbf{Z} by a constant. So, according to [2, Theorem 6.1.15], there exist an ultrafilter U and an isomorphism $f: \mathbf{Z}^U \rightarrow \widehat{\mathbf{Z}}^U$ which fixes the elements of \mathbf{Z} . We define an isomorphism $\bar{f}: G^U \rightarrow \widehat{G}^U$ which fixes the elements of G by $\bar{f}(y_a z_1^{a(1)} \cdots z_s^{a(s)}) = y_a z_1^{f(a(1))} \cdots z_s^{f(a(s))}$ for $1 \leq a \leq r$ and $a(1), \dots, a(s) \in \mathbf{Z}^U$.

2. Proof of Theorem 2. First, we reduce ourselves to the case of finitely generated torsion-free nilpotent groups:

If G is a finitely generated nilpotent-by-finite group, then, by Lemma 1.1, there exists an integer $p \geq 1$ such that G^p is nilpotent and torsion-free. We have $\widehat{G}^p = \widehat{G^p}$ according to [9, Lemma 3, p. 223].

Proposition 2.1 of [6] implies the existence of an integer $q \geq 1$ such that each element of G^p can be written $x = x_1^p \cdots x_q^p$ with $x_1, \dots, x_q \in G$. The group G satisfies the positive sentence $(\forall u_1 \cdots \forall u_{q+1})(\exists v_1 \cdots \exists v_q)(u_1^p \cdots u_{q+1}^p = v_1^p \cdots v_q^p)$. This sentence is also true in \widehat{G} since, by [6, Theorem 3.3], \widehat{G} is an image of an elementary extension of G . So, the formula $\theta(x) = (\exists u_1 \cdots \exists u_q)(x = u_1^p \cdots u_q^p)$ defines G^p in G and \widehat{G}^p in \widehat{G} .

To any existential sentence

$$\varphi = (\exists v_1 \cdots \exists v_n)\psi(v_1, \dots, v_n),$$

where ψ is a quantifier-free formula, we associate the sentence

$$\bar{\varphi} = (\exists v_1 \cdots \exists v_n)(\theta(v_1) \wedge \cdots \wedge \theta(v_n) \wedge \psi(v_1, \dots, v_n))$$

which is equivalent to an existential sentence. The sentence $\bar{\varphi}$ is false in G and true in \widehat{G} as soon as φ is false in G^p and true in $\widehat{G}^p = \widehat{G^p}$.

From now on, we suppose G nilpotent and torsion-free. Then, Theorem 2 is proved through a series of lemmas:

LEMMA 2.1. *In G , there are elements x, y, z such that $[x, y] = z \neq 1$, $[x, z] = 1$ and $[y, z] = 1$.*

PROOF. As G is nilpotent and nonabelian,

$$Z_1(G) = \{x \in G \mid [x, y] = 1 \text{ for each } y \in G\}$$

is strictly contained in

$$Z_2(G) = \{x \in G \mid [x, y] \in Z_1(G) \text{ for each } y \in G\}.$$

For each element $x \in Z_2(G) - Z_1(G)$, there exists an element $y \in G$ such that $[x, y] \neq 1$. We have $[x, z] = 1$ and $[y, z] = 1$ for $z = [x, y]$.

LEMMA 2.2. *Let x, y, z be elements of G such that $[x, y] = z \neq 1$, $[x, z] = 1$ and $[y, z] = 1$. Denote by N the group which is obtained by equipping the set $\widehat{\mathbf{Z}}^3$ with the product $(a_1, b_1, c_1)(a_2, b_2, c_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2 - b_1a_2)$. Then, the map $(a, b, c) \rightarrow x^a y^b z^c$ is an injective homomorphism from N to \widehat{G} .*

PROOF. N can be identified with the profinite completion of the nilpotent group M which is obtained by equipping \mathbf{Z}^3 with the product $(a_1, b_1, c_1)(a_2, b_2, c_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2 - b_1a_2)$. The map $(a, b, c) \rightarrow x^a y^b z^c$ is a continuous homomorphism from N to \widehat{G} . The restriction of this map to M is an injective homomorphism from M to G . So, Lemma 2.2 follows from [9, Proposition 3, (iii), p. 222].

LEMMA 2.3. *For each set P of prime numbers, there exists an element $a \in \widehat{\mathbf{Z}}$ such that, for each integer $n \geq 1$, $a \in n\widehat{\mathbf{Z}}$ if and only if n has no divisor in P .*

PROOF. Consider, for each integer $n \geq 1$, the set E_n of all elements $x \in \widehat{\mathbf{Z}}$ such that, for each integer $m \leq n$, x belongs to $m\widehat{\mathbf{Z}}$ if and only if m has no divisor in P . The set E_n is nonempty since it contains the product of the integers $m \leq n$ which have no divisor in P . It is closed in $\widehat{\mathbf{Z}}$ as a—necessarily finite—union of classes modulo $n\widehat{\mathbf{Z}}$.

So, $(E_n)_{n \in \mathbf{N}^*}$ is a decreasing sequence of nonempty closed subsets in the compact space $\widehat{\mathbf{Z}}$, and the intersection of this sequence is also nonempty. If a is an element of this intersection, then, for each integer $n \geq 1$, a belongs to $n\widehat{\mathbf{Z}}$ if and only if n has no divisor in P . This completes the proof of Lemma 2.3.

In a group, the equalities $[x, y] = 1$ and $[x, y] = z$ are respectively equivalent to the atomic formulas $xy = yx$ and $xy = yxz$. So, the following two lemmas imply the existence of existential sentences built up from the multiplication symbol which are false in G and true in \widehat{G} :

LEMMA 2.4. *For each integer $n \geq 1$, there exist elements $x_1, \dots, x_{3n} \in \widehat{G}$ such that*

- (1) $[x_i, x_{i+n}] = x_{i+2n} \neq 1$ for $1 \leq i \leq n$;
- (2) $[x_i, x_j] = 1$ for any integers $1 \leq i \leq j \leq 3n$ such that $i \geq n + 1$ or $j - i \neq n$.

PROOF. We consider elements $x, y, z \in G$ such that $[x, y] = z \neq 1$, $[x, z] = 1$ and $[y, z] = 1$, a partition $P = P_1 \cup \dots \cup P_n$ of the set P of all prime numbers into n nonempty subsets and, for each integer $1 \leq i \leq n$, an element $a(i) \in \widehat{\mathbf{Z}}$ such that, for each integer $q \geq 1$, $a(i)$ belongs to $q\widehat{\mathbf{Z}}$ if and only if q has no prime divisor in P_i . We write $x_i = x^{a(i)}$, $x_{i+n} = y^{a(i)}$ and $x_{i+2n} = z^{a(i)^2}$ for each integer $1 \leq i \leq n$. Then, we use Lemma 2.2 to prove that x_1, \dots, x_{3n} satisfy properties (1) and (2) of Lemma 2.4.

For each integer $1 \leq i \leq n$, we have

$$[x_i, x_{i+n}] = [x^{a(i)}, y^{a(i)}] = z^{a(i)^2} = x_{i+2n} \neq 1.$$

For any integers $1 \leq i, j \leq n$ such that $i \neq j$, we have $a(i)a(j) = 0$ since $a(i)a(j)$ belongs to $q\widehat{\mathbf{Z}}$ for each integer $q \geq 1$; it follows

$$[x_i, x_{j+n}] = [x^{a(i)}, y^{a(j)}] = z^{a(i)a(j)} = 1.$$

Finally, for any integers $1 \leq i, j \leq n$, we have

$$\begin{aligned} [x_i, x_j] &= [x^{a(i)}, x^{a(j)}] = 1, & [x_{i+n}, x_{j+n}] &= [y^{a(i)}, y^{a(j)}] = 1, \\ [x_{i+2n}, x_{j+2n}] &= [z^{a(i)}, z^{a(j)}] = 1, & [x_i, x_{j+2n}] &= [x^{a(i)}, z^{a(j)}] = 1, \\ & & [x_{i+n}, x_{j+2n}] &= [y^{a(i)}, z^{a(j)}] = 1. \end{aligned}$$

DEFINITION. For each finitely generated nilpotent group M , there exists a series of subgroups $\{1\} = M_0 \subset M_1 \subset \dots \subset M_n = M$ with M_{i-1} normal in M_i and M_i/M_{i-1} cyclic for $1 \leq i \leq n$. The number of integers i such that M_i/M_{i-1} is infinite does not depend on the choice of the series. It is called *Hirsch number* of M , according to [9, p. 16].

LEMMA 2.5. For each integer n which is larger than the Hirsch number of G , there are no elements $x_1, \dots, x_{3n} \in G$ which satisfy properties (1) and (2) of Lemma 2.4.

PROOF. Let us consider elements $x_1, \dots, x_{3n} \in G$ which satisfy these properties. The subgroup of G which is generated by x_1, \dots, x_n is abelian. For any integers $a(1), \dots, a(n) \in \mathbf{Z}$, we have $x_1^{a(1)} \dots x_n^{a(n)} = 1$ if and only if $a(1) = \dots = a(n) = 0$. In fact, if $x_1^{a(1)} \dots x_n^{a(n)}$ is equal to 1, then, for each integer $1 \leq i \leq n$, we have $1 = [x_1^{a(1)} \dots x_n^{a(n)}, x_{i+n}] = x_{i+2n}^{a(i)}$ and, therefore, $a(i) = 0$. So, x_1, \dots, x_n generate a free abelian subgroup which has rank n as a \mathbf{Z} -module. According to [9, p. 16], this is impossible if n is larger than the Hirsch number of G .

3. Proof of the proposition. The groups M, N are elementarily equivalent according to [1, Corollary] and [6, Theorem 5.5]. Theorem 6.1.15 of [2] implies the existence of an ultrafilter U such that M^U and N^U are isomorphic. We have $G^U = \langle M^U \times M^U, y; y^2 = 1, y^{-1}(u, v)y = (v, u) \text{ for any } u, v \in M^U \rangle$ and $H^U = \langle N^U \times N^U, z; z^2 = 1, z^{-1}(u, v)z = (v, u) \text{ for any } u, v \in N^U \rangle$. To any isomorphism $\varphi: M^U \rightarrow N^U$, we can associate an isomorphism $\Phi: G^U \rightarrow H^U$ defined by $\Phi(y) = z$ and $\Phi(u, v) = (\varphi(u), \varphi(v))$ for any $u, v \in M^U$. Consequently, the elementary equivalence of G and H follows from [2, Theorem 4.1.9].

It remains to prove that $G \times \mathbf{Z}$ and $H \times \mathbf{Z}$ are not isomorphic. For this part of the proof, we write $g(u, v) = (v, u)$ for any $u, v \in M$ and $h(u, v) = (v, u)$ for any $u, v \in N$. If x_1, \dots, x_n are elements of a group A , we denote by $\langle x_1, \dots, x_n \rangle$ the subgroup of A which is generated by x_1, \dots, x_n .

LEMMA 3.1. We have $[G, G] = \langle (b, b^{-1}), (a, a^{-1}), (a^5, 1) \rangle$ and $[H, H] = \langle (d, d^{-1}), (c, c^{-1}), (c^5, 1) \rangle$.

PROOF OF LEMMA 3.1 (FOR G). The subgroup $[G, G]$ is generated by the elements:

$$\begin{aligned} & [u, v] \quad \text{for } u, v \in M \times M, \\ [u, yv] &= u^{-1}v^{-1}y^{-1}uyv = u^{-1}v^{-1}g(u)v = u^{-1}g(u)g(u)^{-1}v^{-1}g(u)v \\ &= (u^{-1}g(u))[g(u), v] \quad \text{for } u, v \in M \times M, \\ [yu, yv] &= u^{-1}y^{-1}v^{-1}y^{-1}yuyv = u^{-1}y^{-1}v^{-1}uyv = u^{-1}g(v^{-1}u)v \\ &= u^{-1}g(v)^{-1}g(u)v = u^{-1}g(u)g(u)^{-1}g(v)^{-1}g(u)g(v)g(v)^{-1}v \\ &= (u^{-1}g(u))[g(u), g(v)](v^{-1}g(v))^{-1} \quad \text{for } u, v \in M \times M. \end{aligned}$$

So, $[G, G]$ is generated by $[M \times M, M \times M] = [M, M] \times [M, M] = \langle a^5 \rangle \times \langle a^5 \rangle$ and the elements $x^{-1}g(x)$ for $x \in M \times M$. Moreover, we have

$$(u, v)^{-1}g(u, v) = (u^{-1}, v^{-1})(v, u) = (u^{-1}v, v^{-1}u) = (u^{-1}v, (u^{-1}v)^{-1})$$

for any $u, v \in M$ and $(u, u^{-1}) = (1, u)^{-1}g(1, u)$ for each $u \in M$. So, an element of G can be written $x^{-1}g(x)$ with $x \in M \times M$ if and only if it can be put in the form (u, u^{-1}) with $u \in M$.

PROOF OF THE PROPOSITION (END). $K = \langle (a, a^{-1}), (a^5, 1) \rangle$ (respectively $L = \langle (c, c^{-1}), (c^5, 1) \rangle$) is the set of all elements $x \in [G, G]$ (respectively $x \in [H, H]$) such that $x^{25} = 1$; $[G, G]/K$ and $[H, H]/L$ are isomorphic to \mathbf{Z} . If f is an isomorphism from $G \times \mathbf{Z}$ to $H \times \mathbf{Z}$, then the restriction of f to $[G, G] = [G \times \mathbf{Z}, G \times \mathbf{Z}]$ is an isomorphism from $[G, G]$ to $[H, H] = [H \times \mathbf{Z}, H \times \mathbf{Z}]$; the restriction of f to K is an isomorphism from K to L and f induces an isomorphism from $[G, G]/K$ to $[H, H]/L$. So, we have $f(b, b^{-1}) = (du, d^{-1}v)$ or $f(b, b^{-1}) = (d^{-1}u, dv)$ with $u, v \in \langle c \rangle$.

Moreover, we have $(a, 1)^5 \neq 1$ and $(a, 1)^{25} = 1$, whence $f(a, 1)^5 \neq 1$ and $f(a, 1)^{25} = 1$, and therefore $f(a, 1) = (c^m, c^n)$ with m or n not divisible by 5. The equality $(b, b^{-1})^{-1}(a, 1)(b, b^{-1}) = (a, 1)^6$ implies $f(b, b^{-1})^{-1}(c^m, c^n)f(b, b^{-1}) = (c^{6m}, c^{6n})$. The hypotheses $f(b, b^{-1}) = (du, d^{-1}v)$ and $f(b, b^{-1}) = (d^{-1}u, dv)$ with $u, v \in \langle c \rangle$ respectively imply

$$\begin{aligned} (c^{6m}, c^{6n}) &= (du, d^{-1}v)^{-1}(c^m, c^n)(du, d^{-1}v) = (u^{-1}d^{-1}c^m du, v^{-1}dc^nd^{-1}v) \\ &= (u^{-1}c^{11m}u, v^{-1}c^{16n}v) = (c^{11m}, c^{16n}) \end{aligned}$$

and

$$\begin{aligned} (c^{6m}, c^{6n}) &= (d^{-1}u, dv)^{-1}(c^m, c^n)(d^{-1}u, dv) = (u^{-1}dc^md^{-1}u, v^{-1}d^{-1}c^ndv) \\ &= (u^{-1}c^{16m}u, v^{-1}c^{11n}v) = (c^{16m}, c^{11n}), \end{aligned}$$

whence a contradiction in both cases since m or n is not divisible by 5.

NOTE ADDED IN PROOF (May 10, 1988). Since the completion of this manuscript, we have obtained some generalizations which will appear in *Archiv der Mathematik* under the title *Elementary equivalence of a polycyclic-by-finite group and its profinite completion*. We prove that Theorem 2 is true for polycyclic-by-finite groups and not only for finitely generated nilpotent-by-finite groups. It follows that a polycyclic-by-finite group is elementarily equivalent to its profinite completion if and only if it is abelian-by-finite.

REFERENCES

1. G. Baumslag, *Residually finite groups with the same finite images*, *Compositio Math.* **29** (1974), 249–252.
2. C. C. Chang and H. J. Keisler, *Model theory*, *Studies in Logic* no. 73, North-Holland, Amsterdam, 1973.
3. R. Hirshon, *The cancellation of an infinite cyclic group in direct products*, *Arch. Math.* **26** (1975), 134–138.
4. —, *Some cancellation theorems with applications to nilpotent groups*, *J. Austral. Math. Soc. Ser. A* **23** (1977), 147–165.
5. —, *The equivalence of $\times^t C \cong \times^t D$ and $J \times C \cong J \times D$* , *Trans. Amer. Math. Soc.* **249** (1979), 331–340.

6. F. Oger, *Equivalence élémentaire entre groupes finis-par-abéliens de type fini*, Comment. Math. Helv. **57** (1982), 469–480.
7. ———, *The model theory of finitely generated finite-by-abelian groups*, J. Symbolic Logic **49** (1984), 1115–1124.
8. D. Robinson, *Finiteness conditions and generalized soluble groups*, t. 1, Ergeb. Math. Grenzgeb., vol. 62, Springer-Verlag, Berlin and New York, 1972.
9. D. Segal, *Polycyclic groups*, Cambridge Univ. Press, Cambridge, 1983.
10. R. B. Warfield, Jr., *Nilpotent groups*, Lecture Notes in Math., vol. 513, Springer-Verlag, Berlin and New York, 1976.
11. ———, *Genus and cancellation for groups with finite commutator subgroup*, J. Pure Appl. Algebra **6** (1975), 125–132.

DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ PARIS VII, 75.251 PARIS CEDEX 05,
FRANCE