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FINITE EXTENSIONS OF RINGS

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ABSTRACT. The paper concerns some cases of ring extensions $R \subset S$, where S is finitely generated as a right R-module and R is right Noetherian. In §1 it is shown that if R is a Jacobson ring, then so is S, with the converse true in the PI case. In §2 we show that if S is semiprime PI, R must also be left (as well as right) Noetherian and S is finitely generated as a left R-module. §3 contains a result on E-rings.

In this paper we collect some results concerning the relationship between rings $R \subset S$, where S is finitely generated as a right R-module. For the special cases in which the finite extension S of R is normalizing or centralizing, many theorems have been proved. In this paper we obtain some results of a more general nature.

1. Jacobson property. We call a ring R a Jacobson ring if every prime ideal of R is an intersection of primitive ideals. Let $J(R)$ denote the Jacobson radical of R and $N(R)$ the lower nilradical, that is, the intersection of all the prime ideals of R. With this notation, the alternative formulations of the Jacobson property are:

$J(R/P) = 0$ for all prime ideals P of R.

$J(\overline{R}) = N(\overline{R})$ for every homomorphic image $\overline{R}$ of R.

If R is PI or right Noetherian, then every nil ideal is contained in $N(R)$, and the last condition is equivalent to:

$J(\overline{R})$ is nil for every homomorphic image $\overline{R}$ of R.

THEOREM 1. *Let R be a right Noetherian subring of a ring S, such that S is finitely generated as a right R-module. Then, if R is Jacobson, so must S be Jacobson.*

PROOF. Given any prime ideal P of S, we want to show that $J(S/P) = 0$. Since the hypothesis of the theorem holds for the ring embedding $R/P \cap R \subset S/P$, the problem reduces to the case where S is a prime ring. We therefore want to show that, if S is prime, then $J(S) = 0$.

Suppose $J(S) \neq 0$. By a standard result on Goldie rings, $J(S)$ must contain a regular element, say a. Since S_R is finitely generated and R is right Noetherian, there is a positive integer n, such that the elements $1, a, a^2, \ldots, a^n$ are integrally dependent over R. That is, for some $r_{n-1}, \ldots, r_1, r_0 \in R$, we have $a^n + a^{n-1}r_{n-1} + \cdots + ar_1 + r_0 = 0$. If n is minimal, then, since a is regular, $r_0 \neq 0$. But then $r_0 \in J(S) \cap R$. Hence $J(S) \cap R \neq 0$.

We claim that $J(S) \cap R \subset J(R)$. Let $x \in J(S) \cap R$ and suppose $x \notin J(R)$. Then there is an element $r \in R$ for which $1 - rx$ is not invertible in R. But $1 - rx$ is

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invertible in S, say $(1-rx)s = 1$ for some $s \in S$. Since s is integral over R we have $s^k + s^{k-1}a_{k-1} + \cdots + sa_1 + a_0 = 0$ for some $a_0, a_1, \ldots, a_{k-1} \in R$. Multiplying this equation by $(1-rx)^{k-1}$ and solving for s yields:

$$s = -[a_{k-1} + (1-rx)a_{k-2} + \cdots + (1-rx)^{k-1}a_0] \in R,$$

which is a contradiction. Consequently, $J(S) \cap R \subset J(R)$.

Because R is right Noetherian and Jacobson, we have that $J(R) = N(R)$, the maximal nilpotent ideal of R. Hence $J(S) \cap R$ is a nonzero nilpotent ideal of R.

Consider the ring embedding $R/J(S) \cap R \subset S/J(S)$. This is a finite ring extension. Invoking some results on Krull dimension (as defined in [**6**]), we shall obtain a contradiction. We use notation $|R|$ for Krull dimension of the ring R, and $|M|_R$ for Krull dimension of the R-module M.

On one hand, since $J(S) \neq 0$ and S is prime, $J(S)$ contains a regular element and therefore $|S/J(S)|_R < |S|_R = |R|$.

On the other hand, since $J(S) \cap R$ is nilpotent, we have:

$$|R| = |R/J(S) \cap R| = |S/J(S)|_{R/J(S) \cap R} = |S/J(S)|_R,$$

which contradicts the inequality above.

The proof is now complete, since the assumption $J(S) \neq 0$ led to a contradiction. $\square$

In the PI case, the converse of Theorem 1 holds. To prove this, note that the Jacobson property goes up and down for the following two classes of ring extensions. First, if S is a liberal extension of R, that is, if S is finitely generated as a module over R by elements that centralize R, then S is Jacobson if and only if R is Jacobson (Robson and Small [**8**]). The second type of extensions that we need is considered in the following lemma, half of which follows immediately from Blair's results in [**3**].

LEMMA. *Suppose R is a central subring of a PI ring S and that S is integral over R. Then S is Jacobson if and only if R is Jacobson.*

PROOF. Blair [**3**] proved that, if S is integral over its subring R, for every prime ideal P of R, there is a prime ideal Q of S with $Q \cap R = P$. Moreover, $J(R) = J(S) \cap R$.

If S is a Jacobson ring, let P be a prime ideal of R and Q a prime of S lying over P. The hypothesis of the lemma carries over to the ring extension $R/P \subset S/Q$ and we have $J(R/P) = J(S/Q) \cap R/P = 0$. Since P was an arbitrary prime ideal of R, R is Jacobson. This part of the lemma did not require the hypothesis that S be PI.

Conversely, let R be Jacobson. To show that S is Jacobson, it is enough to prove that $J(S/Q)$ is nil for every ideal Q of S. Since the hypothesis of the lemma holds for the ring embedding $R/Q \cap R \subset S/Q$, it is enough to show that if R is Jacobson, then $J(S)$ is nil. Let $x \in J(S)$. $R\langle x\rangle$, the ring generated by R and x, is a Jacobson ring, since it is a homomorphic image of the polynomial ring $R[X]$. As in the proof of Theorem 1, integrality of $R\langle x\rangle$ over R gives $J(S) \cap R\langle x\rangle \subset J(R\langle x\rangle)$. Consequently, $x \in J(R\langle x\rangle)$ and x is nilpotent. Thus $J(S)$ is nil. $\square$

THEOREM 2. *Let $R \subset S$ be rings such that R is right Noetherian, S is PI and finitely generated as a right R-module. Then S is Jacobson if and only if R is Jacobson.*

PROOF. In view of Theorem 1, it is enough to show that "S Jacobson" implies "R Jacobson".

If P is a prime ideal of R, there is a prime ideal P' of S for which $P' \cap R \subset P$ (take P' to be maximal among all ideals I of S for with $I \cap R \subset P$). Now consider the ring embedding of $R/P' \cap R$ in S/P'. If $R/P' \cap R$ is Jacobson, then so is R/P. Thus one can replace S by S/P' and assume that S is prime.

By Posner's theorem, S has a classical ring of quotients Q, which is a finite-dimensional central simple algebra, $Q = M_n(D)$, where D is a division ring finite-dimensional over its center. If Z is the center of D and K a maximal subfield of D, then we have $S \subset Q \subset M_n(D) \otimes_Z K = M_m(K)$ for some m. Thus every element of S can be considered as a matrix and has a (reduced) characteristic polynomial with coefficients in Z. Let T denote the subalgebra of $M_m(K)$ generated by coefficients of these polynomials and let TS denote the subalgebra of $M_n(D)$ generated by T and S, TR the subalgebra generated by T and R. (For the construction and properties of the reduced trace ring TS, see [**10** and **1**].)

Consider the diagram

$$\begin{tikzcd}
 & & TS \arrow[dll, dash] \arrow[d, dash] \arrow[dr, dash] & \\
TR \arrow[rr, dash] \arrow[drr, dash] & & T & S \arrow[dl, dash] \\
 & & R &
\end{tikzcd}$$

The path we take in proving the implication "S Jacobson $\Rightarrow$ R Jacobson" is indicated by bold line segments. Note that all rings in the diagram are PI.

It is well known that, if S is right Notherian, then TS is a finite right S-module [**1**]. By Theorem 1, TS is Jacobson. It is also well known that TS is integral over its center [**1**], and therefore also over T.

By the lemma above, T must be Jacobson. TR is integral over T (as a subring of TS), and so is Jacobson. Since TS is a finite S-module, it is also finitely generated over R, as is its R-submodule TR. Thus TR is a liberal extension of R, and consequently R is Jacobson. This completes the proof. □

2. Changing sides. We shall be concerned now with some situations in which a right-hand property of a ring can be transferred to a left-hand one. The results we obtain will be of a going-down type. We cite two results which will be used in the proofs. The first of these results was proved by Cauchon [**4**] for the prime case. To extend this to the semiprime case is routine (see [**9**, pp. 225–226]). The result is

(i) A semiprime PI ring which satisfies ACC on ideals is right (and left) Noetherian.

The second result is due to Björk [**2**].

(ii) If R is a right Noetherian subring of a right artinian ring S such that S is finitely generated as a right R-module, then R is right artinian.

THEOREM 3. *Let S be a semiprime PI ring and R a right Noetherian subring of S such that S is a finitely generated right R-module. Then S is finitely generated as a left R-module and R is left Noetherian.*

PROOF. Consider the inclusion of rings: $R[x] \subset R + xS[x] \subset S[x]$. Since S_R is finitely generated, S is right Noetherian. Therefore $R[x]$ and $S[x]$ are also right Noetherian, as is the finitely generated $R[x]$-module $R + xS[x]$. Since S has no nonzero nil ideals, $S[x]$ is semiprime. Because $S[x]$ is a PI ring, so is its subring $R + xS[x]$. $R + xS[x]$ is also semiprime, since it contains an ideal of a semiprime ring.

By the cited result (i) above, $R + xS[x]$ is left (as well as right) Noetherian. But then R, as a homomorphic image of $R + xS[x]$ must also be Noetherian.

The ideal $xS[x]$ of $R + xS[x]$ is finitely generated on the left, since $R + xS[x]$ is left Noetherian. Its homomorphic image $xS[x]/x^2S[x]$ has the same $R + xS[x]$-module structure as the R-module structure of S. Thus S is a finitely generated left R-module. □

COROLLARY. *Let S be a primitive PI ring and R a right Noetherian subring of S such that S is a finitely generated right R-module. Then R is left and right artinian and S is finitely generated as a left R-module.*

PROOF. By a famous theorem of Kaplansky, S is a simple algebra, finite-dimensional over its center. By Theorem 3, R is both left and right Noetherian and S is a finitely generated left R-module. By the cited result (ii) of Björk we get that R is left and right artinian. □

The following example shows that the condition that S be a PI ring is necessary.

P. M. Cohn [**5**] has constructed a pair of division rings $D' \subset D$ such that D is a finite-dimensional right vector space over D' but is infinite-dimensional as a left vector space over D'.

Let

$$R = \begin{bmatrix} D' & D \\ 0 & D \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} D & D \\ D & D \end{bmatrix}.$$

Thus S is a simple artinian ring and, as is easily checked, is finitely generated both as a left and right R-module. Moreover, R is right Noetherian. However, R is right artinian, but is *not* left artinian (nor left Noetherian).

3. E-rings. A ring S has been defined to be an E-ring [**7**] if its primitive ideals are *coartinian*, that is, if S/P is simple artinian for every primitive ideal P of S. Examples of such E-rings are Noetherian PI rings and, more generally, all fully bounded Noetherian rings.

The following result is interesting in view of Stafford's example in [**11**] of prime Noetherian rings $R \subset S$ and a simple S-module which has an infinite length over R, even though S is finitely generated both as a left and right R-module.

THEOREM 4. *Let S be an E-ring and R a right Noetherian subring of S such that S is finitely generated as a right R-module. Then every simple right S-module has finite length as a right R-module. Hence every right S-module of finite length has finite length as a right R-module.*

PROOF. Let M be a simple right S-module and set $A = \mathrm{Ann}_S M$, the annihilator of M in S. Then A is a primitive ideal of S, and $R/A \cap R$ is a subring of the simple artinian ring S/A. Clearly S/A is finitely generated as a right $R/A \cap R$-module. By Björk's result, $R/A \cap R$ is right artinian. Since M is finitely generated over $R/A \cap R$, M has finite length as an R-module. $\square$

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