ABSTRACT. The paper concerns some cases of ring extensions $R \subseteq S$, where
$S$ is finitely generated as a right $R$-module and $R$ is right Noetherian. In §1
it is shown that if $R$ is a Jacobson ring, then so is $S$, with the converse true
in the PI case. In §2 we show that if $S$ is semiprime PI, $R$ must also be left
(as well as right) Noetherian and $S$ is finitely generated as a left $R$-module.
§3 contains a result on $E$-rings.

In this paper we collect some results concerning the relationship between rings
$R \subseteq S$, where $S$ is finitely generated as a right $R$-module. For the special cases
in which the finite extension $S$ of $R$ is normalizing or centralizing, many theorems
have been proved. In this paper we obtain some results of a more general nature.

1. Jacobson property. We call a ring $R$ a Jacobson ring if every prime ideal
of $R$ is an intersection of primitive ideals. Let $J(R)$ denote the Jacobson radical of
$R$ and $N(R)$ the lower nilradical, that is, the intersection of all the prime ideals of
$R$. With this notation, the alternative formulations of the Jacobson property are:

$J(R/P) = 0$ for all prime ideals $P$ of $R$.

$J(R) = N(R)$ for every homomorphic image $R$ of $R$.

If $R$ is PI or right Noetherian, then every nil ideal is contained in $N(R)$, and the
last condition is equivalent to:

$J(R)$ is nil for every homomorphic image $R$ of $R$.

THEOREM 1. Let $R$ be a right Noetherian subring of a ring $S$, such that $S$
is finitely generated as a right $R$-module. Then, if $R$ is Jacobson, so must $S$ be
Jacobson.

PROOF. Given any prime ideal $P$ of $S$, we want to show that $J(S/P) = 0$. Since
the hypothesis of the theorem holds for the ring embedding $R/P \cap R \subseteq S/P$, the
problem reduces to the case where $S$ is a prime ring. We therefore want to show
that, if $S$ is prime, then $J(S) = 0$.

Suppose $J(S) \neq 0$. By a standard result on Goldie rings, $J(S)$ must contain a
regular element, say $a$. Since $S_R$ is finitely generated and $R$ is right Noetherian,
there is a positive integer $n$, such that the elements $1, a, a^2, \ldots, a^n$ are integrally
dependent over $R$. That is, for some $r_{n-1}, \ldots, r_1, r_0 \in R$, we have $a^n + a^{n-1}r_{n-1} +
\cdots + ar_1 + r_0 = 0$. If $n$ is minimal, then, since $a$ is regular, $r_0 \neq 0$. But then
$r_0 \in J(S) \cap R$. Hence $J(S) \cap R \neq 0$.

We claim that $J(S) \cap R \subseteq J(R)$. Let $x \in J(S) \cap R$ and suppose $x \notin J(R)$.
Then there is an element $r \in R$ for which $1 - rx$ is not invertible in $R$. But $1 - rx$ is

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invertible in $S$, say $(1 - rx)s = 1$ for some $s \in S$. Since $s$ is integral over $R$ we have $s^k + s^{k-1}a_{k-1} + \cdots + sa_1 + a_0 = 0$ for some $a_0, a_1, \ldots, a_{k-1} \in R$. Multiplying this equation by $(1 - rx)^{k-1}$ and solving for $s$ yields:

$$s = -[a_{k-1} + (1 - rx)a_{k-2} + \cdots + (1 - rx)^{k-1}a_0] \in R,$$

which is a contradiction. Consequently, $J(S) \cap R \subset J(R)$.

Because $R$ is right Noetherian and Jacobson, we have that $J(R) = N(R)$, the maximal nilpotent ideal of $R$. Hence $J(S) \cap R$ is a nonzero nilpotent ideal of $R$.

Consider the ring embedding $R/J(S) \cap R \subset S/J(S)$. This is a finite ring extension. Invoking some results on Krull dimension (as defined in [6]), we shall obtain a contradiction. We use notation $|R|$ for Krull dimension of the ring $R$, and $|M|_R$ for Krull dimension of the $R$-module $M$.

On one hand, since $J(S) \neq 0$ and $S$ is prime, $J(S)$ contains a regular element and therefore $|S/J(S)|_R < |S|_R = |R|$.

On the other hand, since $J(S) \cap R$ is nilpotent, we have:

$$|R| = |R/J(S) \cap R| = |S/J(S)|_{R/J(S) \cap R} = |S/J(S)|_R,$$

which contradicts the inequality above.

The proof is now complete, since the assumption $J(S) \neq 0$ led to a contradiction.

In the PI case, the converse of Theorem 1 holds. To prove this, note that the Jacobson property goes up and down for the following two classes of ring extensions. First, if $S$ is a liberal extension of $R$, that is, if $S$ is finitely generated as a module over $R$ by elements that centralize $R$, then $S$ is Jacobson if and only if $R$ is Jacobson (Robson and Small [8]). The second type of extensions that we need is considered in the following lemma, half of which follows immediately from Blair's results in [3].

**Lemma.** Suppose $R$ is a central subring of a PI ring $S$ and that $S$ is integral over $R$. Then $S$ is Jacobson if and only if $R$ is Jacobson.

**Proof.** Blair [3] proved that, if $S$ is integral over its subring $R$, for every prime ideal $P$ of $R$, there is a prime ideal $Q$ of $S$ with $Q \cap R = P$. Moreover, $J(R) = J(S) \cap R$.

If $S$ is a Jacobson ring, let $P$ be a prime ideal of $R$ and $Q$ a prime of $S$ lying over $P$. The hypothesis of the lemma carries over to the ring extension $R/P \subset S/Q$ and we have $J(R/P) = J(S/Q) \cap R/P = 0$. Since $P$ was an arbitrary prime ideal of $R$, $R$ is Jacobson. This part of the lemma did not require the hypothesis that $S$ be PI.

Conversely, let $R$ be Jacobson. To show that $S$ is Jacobson, it is enough to prove that $J(S/Q)$ is nil for every ideal $Q$ of $S$. Since the hypothesis of the lemma holds for the ring embedding $R/Q \cap R \subset S/Q$, it is enough to show that if $R$ is Jacobson, then $J(S)$ is nil. Let $x \in J(S)$. $R(x)$, the ring generated by $R$ and $x$, is a Jacobson ring, since it is a homomorphic image of the polynomial ring $R[X]$. As in the proof of Theorem 1, integrality of $R(x)$ over $R$ gives $J(S) \cap R(x) \subset J(R(x))$. Consequently, $x \in J(R(x))$ and $x$ is nilpotent. Thus $J(S)$ is nil. □
THEOREM 2. Let \( R \subset S \) be rings such that \( R \) is right Noetherian, \( S \) is PI and finitely generated as a right \( R \)-module. Then \( S \) is Jacobson if and only if \( R \) is Jacobson.

PROOF. In view of Theorem 1, it is enough to show that “\( S \) Jacobson” implies “\( R \) Jacobson”.

If \( P \) is a prime ideal of \( R \), there is a prime ideal \( P' \) of \( S \) for which \( P' \cap R \subset P \) (take \( P' \) to be maximal among all ideals \( I \) of \( S \) for with \( I \cap R \subset P \)). Now consider the ring embedding of \( R/P' \cap R \subset S/P' \). If \( R/P' \cap R \) is Jacobson, then so is \( R/P \). Thus one can replace \( S \) by \( S/P' \) and assume that \( S \) is prime.

By Posner’s theorem, \( S \) has a classical ring of quotients \( Q \), which is a finite-dimensional central simple algebra, \( Q = M_n(D) \), where \( D \) is a division ring finite-dimensional over its center. If \( Z \) is the center of \( D \) and \( K \) a maximal subfield of \( D \), then we have \( S \subset Q \subset M_n(D) \otimes_Z K = M_m(K) \) for some \( m \). Thus every element of \( S \) can be considered as a matrix and has a (reduced) characteristic polynomial with coefficients in \( Z \). Let \( T \) denote the subalgebra of \( M_m(K) \) generated by coefficients of these polynomials and let \( TS \) denote the subalgebra of \( M_n(D) \) generated by \( T \) and \( S \), \( TR \) the subalgebra generated by \( T \) and \( R \). (For the construction and properties of the reduced trace ring \( TS \), see [10 and 1].)

Consider the diagram

\[
\begin{array}{ccc}
TR & & TS \\
\downarrow & & \downarrow \\
R & & S \\
\end{array}
\]

The path we take in proving the implication “\( S \) Jacobson \( \Rightarrow \) \( R \) Jacobson” is indicated by bold line segments. Note that all rings in the diagram are PI.

It is well known that, if \( S \) is right Noetherian, then \( TS \) is a finite right \( S \)-module [1]. By Theorem 1, \( TS \) is Jacobson. It is also well known that \( TS \) is integral over its center [1], and therefore also over \( T \).

By the lemma above, \( T \) must be Jacobson. \( TR \) is integral over \( T \) (as a subring of \( TS \)), and so is Jacobson. Since \( TS \) is a finite \( S \)-module, it is also finitely generated over \( R \), as is its \( R \)-submodule \( TR \). Thus \( TR \) is a liberal extension of \( R \), and consequently \( R \) is Jacobson. This completes the proof. \( \Box \)

2. Changing sides. We shall be concerned now with some situations in which a right-hand property of a ring can be transferred to a left-hand one. The results we obtain will be of a going-down type. We cite two results which will be used in the proofs. The first of these results was proved by Cauchon [4] for the prime case. To extend this to the semiprime case is routine (see [9, pp. 225–226]). The result is

(i) A semiprime PI ring which satisfies ACC on ideals is right (and left) Noetherian.

The second result is due to Björk [2].

(ii) If \( R \) is a right Noetherian subring of a right artinian ring \( S \) such that \( S \) is finitely generated as a right \( R \)-module, then \( R \) is right artinian.
**THEOREM 3.** Let $S$ be a semiprime PI ring and $R$ a right Noetherian subring of $S$ such that $S$ is a finitely generated right $R$-module. Then $S$ is finitely generated as a left $R$-module and $R$ is left Noetherian.

**PROOF.** Consider the inclusion of rings: $R[x] \subset R + xS[x] \subset S[x]$. Since $S_R$ is finitely generated, $S$ is right Noetherian. Therefore $R[x]$ and $S[x]$ are also right Noetherian, as is the finitely generated $R[x]$-module $R + xS[x]$. Since $S$ has no nonzero nil ideals, $S[x]$ is semiprime. Because $S[x]$ is a PI ring, so is its subring $R + xS[x]$. $R + xS[x]$ is also semiprime, since it contains an ideal of a semiprime ring.

By the cited result (i) above, $R + xS[x]$ is left (as well as right) Noetherian. But then $R$, as a homomorphic image of $R + xS[x]$ must also be Noetherian.

The ideal $xS[x]$ of $R + xS[x]$ is finitely generated on the left, since $R + xS[x]$ is left Noetherian. Its homomorphic image $xS[x]/x^2S[x]$ has the same $R + xS[x]$-module structure as the $R$-module structure of $S$. Thus $S$ is a finitely generated left $R$-module. □

**COROLLARY.** Let $S$ be a primitive PI ring and $R$ a right Noetherian subring of $S$ such that $S$ is a finitely generated right $R$-module. Then $R$ is left and right artinian and $S$ is finitely generated as a left $R$-module.

**PROOF.** By a famous theorem of Kaplansky, $S$ is a simple algebra, finite-dimensional over its center. By Theorem 3, $R$ is both left and right Noetherian and $S$ is a finitely generated left $R$-module. By the cited result (ii) of Björk we get that $R$ is left and right artinian. □

The following example shows that the condition that $S$ be a PI ring is necessary.

P. M. Cohn [5] has constructed a pair of division rings $D' \subset D$ such that $D$ is a finite-dimensional right vector space over $D'$ but is infinite-dimensional as a left vector space over $D'$.

Let

$$R = \begin{bmatrix} D' & D \\ 0 & D \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} D & D \\ D & D \end{bmatrix}.$$ 

Thus $S$ is a simple artinian ring and, as is easily checked, is finitely generated both as a left and right $R$-module. Moreover, $R$ is right Noetherian. However, $R$ is right artinian, but is not left artinian (nor left Noetherian).

**3. E-rings.** A ring $S$ has been defined to be an $E$-ring [7] if its primitive ideals are coartinian, that is, if $S/P$ is simple artinian for every primitive ideal $P$ of $S$. Examples of such $E$-rings are Noetherian PI rings and, more generally, all fully bounded Noetherian rings.

The following result is interesting in view of Stafford’s example in [11] of prime Noetherian rings $R \subset S$ and a simple $S$-module which has an infinite length over $R$, even though $S$ is finitely generated both as a left and right $R$-module.

**THEOREM 4.** Let $S$ be an $E$-ring and $R$ a right Noetherian subring of $S$ such that $S$ is finitely generated as a right $R$-module. Then every simple right $S$-module has finite length as a right $R$-module. Hence every right $S$-module of finite length has finite length as a right $R$-module.

**PROOF.** Let $M$ be a simple right $S$-module and set $A = \text{Ann}_S M$, the annihilator of $M$ in $S$. Then $A$ is a primitive ideal of $S$, and $R/A \cap R$ is a subring of the simple
artinian ring $S/A$. Clearly $S/A$ is finitely generated as a right $R/A \cap R$-module. By Björk's result, $R/A \cap R$ is right artinian. Since $M$ is finitely generated over $R/A \cap R$, $M$ has finite length as an $R$-module. □

REFERENCES


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