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AN EXTENSION OF ANDO-KRIEGER'S THEOREM TO ORDERED BANACH SPACES

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ABSTRACT. In this paper it is shown that an operator defined on a suitable ordered Banach space of measurable functions by a positive, irreducible kernel is never quasi-nilpotent, thus giving an extension of Ando-Krieger's theorem for operators defined on ordered Banach spaces.

Roughly speaking, Ando-Krieger's theorem says that positive, irreducible kernel operators on some Banach spaces of measurable functions are never quasi-nilpotent [**6**, V. 6.5, or **10**, Theorem 136.9]. Recent extensions of Ando-Krieger's theorem in the framework of Banach lattices have been given in [**3** and **5**] (see also [**4**, **8**]). It is our purpose in this short note to show that a similar result holds for kernel operators acting in more general ordered vector spaces including Sobolev spaces defined on sufficiently smooth domains of $\mathbf{R}^n$. For an interesting application of this kind of result see, for example, [**1**]. Following H. H. Schaefer [**7**], we always assume that the positive cone of an ordered Banach space is closed. For other definitions and terminology we refer to [**7**].

From now on we assume that (X, Σ, μ) is a finite measure space and denote by $L^0(X, \Sigma, \mu)$, or simply L^0, the set of all μ-measurable functions on (X, Σ, μ). The cone of all positive μ-measurable functions will be denoted by L^0_+. Recall that the family of set $\{V_n : n = 1, 2, 3, \ldots\}$, where $V_n = \{f \in L^0 : \mu(x : |f(x)| \geq n^{-1}) \leq n^{-1}\}$ is a basis of neighborhoods of the origin for the topology of convergence in μ-measure in L^0.

To state Theorem 1 below in a simple way, let us define

DEFINITION. Let $t: X \times X \to \mathbf{R}^+$ be a positive measurable function. We say that t is an irreducible kernel if $\int_{X-S} \int_S t(x,y)\, d\mu(x)\, d\mu(y) > 0$ for all $S \in \Sigma$ such that $\mu(S) > 0$, $\mu(X - S) > 0$.

We give now our extension of Ando-Krieger's theorem. The key for the proof is H. H. Schaefer's approach to the proof of this theorem [**6**, V.6.5 and Lemma V.6.4].

THEOREM 1. *Let H be an ordered Banach space, $H \subseteq L^0$, with a nontrivial positive cone $C = L^0_+ \cap H \neq \{0\}$. Let $T: H \to H$ be a bounded linear operator on H induced by a positive, irreducible kernel $t(\cdot, \cdot)$, i.e. given by*

$$Tf(y) = \int_X t(x,y) f(x)\, d\mu(x), \qquad f \in H,$$

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μ-almost everywhere. Then the spectral radius of T, $r(T)$, is strictly positive. If C is total in H (i.e. $\overline{C-C}=H$) and T is compact, then there exists a strictly positive eigenfunction associated to $r(T)$.

REMARK. Observe that there exists a function $e \in C$ which is strictly positive almost everywhere with respect to μ. In fact, $C \neq \{0\}$, let $f \in C$, $f \neq 0$ and let

$$e\colon = R(\lambda, T)f = \sum_{n=0}^{\infty} \lambda^{-n-1} T^n e \geq 0.$$

Since C is closed in H, $e \in C$ and the strict positivity of e follows from $\lambda e \geq Te$ and the irreducibility of the kernel of T. From now on e denotes the function here constructed.

PROOF. We may assume without loss of generality that $\|T\colon H \to H\| \leq 1$. Let D_T be the natural domain of T, defined by: $D_T := \{u \in L^0\colon \int t(x,y)|u(x)|\,d\mu(x) < \infty \text{ a.e.}\}$ [**9**].

It is clear that T can be extended to a linear map from D_T into L^0. Let Q be the solid hull of $C_1 := \{\mu \in C\colon \|u\|_H \leq 1\}$ i.e. $Q = \{u \in L^0\colon$ there exists $v \in C_1$ such that $|u| \leq v\}$.

If we call the natural extension of T to D_T again by T, then $T(Q) \subseteq Q$. In fact, let $u \in Q$ and let $v \in C_1$ be such that $|u| \leq v$. Then $|Tu| \leq T|u| \leq Tv$. $Tu \in Q$ since $Tv \in C_1$. Let $E_Q = \bigcup_{n=1}^{\infty} nQ$. We can introduce a norm in E_Q using the Minkowski functional of Q. In fact, for $x \in E_Q$ define $\|x\|_Q\colon \inf\{\delta > 0\colon x \in \delta Q\}$. The only thing to prove is that $\|x\|_Q > 0$ for $x \in E_Q$, $x \neq 0$. To do that, observe that Q is a bounded set in L^0. Otherwise, we can find a neighborhood of the origin in L^0, call it V, and a sequence $x_n \in C_1$ such that $x_n \notin 2^n nV$. Let $z = \sum_{n=1}^{\infty} 2^{-n} x_n$. Since $x_n \in C_1$ and C is closed in H, it follows that $z \in C_1 \subseteq Q$ and $z \geq 2^{-n} x_n$. Hence, $z \notin nV$ for all $n \in \mathbf{N}$. This contradiction proves that Q is bounded in L^0. Hence, if $\|x\|_Q = 0$, $x \in n^{-1}Q$ for all $n = 1, 2, \ldots$. Therefore $x = 0$. Assume now that $r(T) = 0$. Let $\lambda > 0$. Using the Neumann series expansion of the resolvent, we see that $R(\lambda, T)e \geq \lambda^{-1}e$. Hence, $\|\lambda^{-1}e\|_Q \leq \|R(\lambda, T)e\|_Q$ for all $\lambda > 0$. Thus, $\lim_{\lambda \to 0^+} \|R(\lambda, T)e\|_Q = +\infty$. Let $\lambda_n \to 0+$ be such that $R(\lambda_n, T)e \notin nQ$ for each $n = 1, 2, \ldots$. Thus, $\|R(\lambda_n, T)e\|_H \geq n$ for $n = 1, 2, \ldots$. We have found a sequence $\lambda_n \to 0+$ such that $\|R(\lambda_n, T)e\|_H \to \infty$. Let $z_n := q_n R(\lambda_n, T)e$ where $q_n := 2^{-n}\|R(\lambda_n, T)e\|_H^{-1}$. Let $z := \sum_{k=1}^{\infty} z_k$. Since C is closed in H, z_n, $z \in C$. Let $y_n := \sum_{k=n}^{\infty} z_k$, $n = 1, 2, \ldots$. Since for all $k \geq n$, $(\lambda_n - T)z_k = (\lambda_n - \lambda_k)z_k + q_n e$, it follows that $Ty_n \leq \lambda_n y_n$, $n = 1, 2, \ldots$. Let $I_z := \{u \in L^0\colon |u| \leq nz$ for some integer $n \geq 1\}$. Let $u \in I_z$ and let $n \geq 1$ be such that $|u| \leq nz$. Then

$$\int_X t(x,y)|u(x)|\,d\mu(x) \leq n \int_X t(x,y)z(x)\,d\mu(x) \leq nTz(y) \leq n\lambda_1 z(y).$$

Define $T_0\colon I_z \to I_z$ by

$$T_0 u(y) = \int t(x,y)u(x)\,d\mu(x), \qquad u \in I_z.$$

We show that $r(T_0) = 0$. For that, observe that for all $n = 1, 2, \ldots$, y_n is a order unit in I_z (i.e. for each $n \in \mathbf{N}$ we can find $k_n > 0$ such that $y_n \leq z \leq k_n y_n$) and $T_0 y_n \leq \lambda_n y_n$. Hence $r(T_0) \leq \lambda_n$, i.e. $r(T_0) = 0$.

It is easy to see that I_z can be identified with the dual of the AL-space $L^1(X, \Sigma, z\,d\mu)$ and T_0 coincides with the adjoint of the operator S_0 on $L^1(X, \Sigma, z\,d\mu)$ given by

$$S_0 v(x) = \int t(x,y)v(y)z(y)\ d\mu(y)$$

$v \in L^1(X, \Sigma, z\,d\mu)$. Since $z(y) > 0$ a.e. and $t(\cdot,\cdot)$ is an irreducible kernel, $t(x,y)z(y)$ is also a positive, irreducible kernel. Hence, S_0 is a positive irreducible kernel operator on $L^1(X, \Sigma, z\,d\mu)$ [**10**, 136.3]. By Ando-Krieger's theorem [**6**, V.6.5, or **10**, 136.9] $r(S_0) > 0$. Therefore $r(T_0) > 0$. A contradiction. Therefore, $r(T) > 0$. The last assertion is a consequence of Krein-Rutman's theorem [**7**, Corollary to 2.4 Appendix]. Remark that any positive eigenfunction $u \in C$ associated to $r(T)$ must be strictly positive a.e. because $t(\cdot,\cdot)$ is an irreducible kernel.

REMARK. We have not been able to prove in Theorem 1 the uniqueness of the positive eigenfunction associated to $r(T)$. This could be done with supplementary information about the cone C or T, available in some concrete problems. This is the case, for example, in [**1**].

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REFERENCES

1. T. Ando and M. Zerner, *Sur une valeur propre d'un opérateur*, Comment. Math. Phys. **93** (1984), 123–139.
2. H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Rev. **18** (1976), 620–709.
3. V. Caselles, *On irreducible operators on Banach lattices*, Indag. Math. **89** (1986), 11–16.
4. J. J. Grobler, *Band irreducible operators*, Indag. Math. **89** (1988), 405–409.
5. B. de Pagter, *Irreducible compact operators*, Math. Z. **192** (1986), 149–153.
6. H. H. Schaefer, *Banach lattices and positive operators*, Springer-Verlag, Berlin, Heidelberg and New York, 1974.
7. ——, *Topological vector spaces*, Macmillan, 1966.
8. ——, *On theorems of de Pagter and Ando-Krieger*, Math. Z. **192** (1986), 155–157.
9. P. Szeptycki, *Notes on integral transformations*, Dissertationes Math. **231** (1984).
10. A. C. Zaanen, *Riesz spaces*. II, North-Holland, 1983.
11. M. Zerner, *Positive operators in Reggeon field theory, why*?, Aspects of Positivity in Functional Analysis (R. Nagel, U. Schlotterback, and M. P. H. Wolff, eds.), North-Holland, 1986, pp. 271–275.

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